On Hopf algebra structures over operads

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Introduction

The observation that many objects possess naturally both a multiplication and a comultiplication, which are compatible, lead to the theory of Hopf algebras (cf. [Abe], [Mon]).

Hopf [Hop41] discovered that a multiplication $M \times M \to M$ and the diagonal $M \to M \times M$ on a manifold M induce a multiplication $H \otimes H \to H$ and a comultiplication $H \to H \otimes H$ on the homology $H_*(M; K)$, and he proved a structure theorem for $H_*(M; K)$.

At first, commutative or cocommutative Hopf algebras were studied. Over a field of characteristic 0, any connected cocommutative Hopf algebra H is of the form $U(\mathrm{Prim}H)$, where $\mathrm{Prim}H$ is the space of primitive elements viewed as a Lie algebra, and where U is the universal enveloping functor. This is the theorem of Milnor and Moore (proven for graded Hopf algebras in [MM65]). Commutative Hopf algebras are K-linear duals of cocommutative Hopf algebras.

Only a few types of examples of noncommutative non-cocommutative Hopf algebras were constructed (cf. [Taf71]), when the study of quantum groups (cf. [Kas]), emerged in the last decades of the twentieth century. Faddeev, Reshetikhin, Takhtajan (cf. [Tak89]) and others obtained quantum groups as noncommutative deformations of algebraic groups. The notion of a quantum group was introduced by Drinfel'd [Dri86]. Quantum groups are given by noncommutative Hopf algebras, the category of quantum groups being opposite to the category of noncommutative Hopf algebras (see [Man]).

Several combinatorial Hopf algebras (see [ABS03]) were discovered, which provided new links between algebra, geometry, combinatorics, and theoretical physics.

For example, noncommutative Hopf algebras of permutations and of quasi-symmetric functions (cf. [MR95]) allow a new approach to the representation theory of symmetric groups.

Connes and Kreimer discovered (cf. [CK98], [Kre98]) that a commutative non-cocommutative Hopf algebra structure on rooted trees encodes renormalization in quantum field theory. The same formulas define a noncommutative Hopf algebra on planar rooted trees. The use of binary planar rooted trees has been proposed by Brouder and Frabetti [BF03] for renormalization.

Recent developments in Hopf algebra theory show that there are important objects

which could be called "non-classical" Hopf algebras. Typical examples are dendriform Hopf algebras [Ron00a, Ron02]. Dendriform algebras are equipped with two operations $\prec$, $\succ$ whose sum is an associative multiplication.

Other examples are magma Hopf algebras (considered in [GH03]), or infinitesimal Hopf algebras (cf. [AS00]). Here the usual Hopf algebra axioms have to be changed.

Instead of the classical types of algebras (like associative algebras, commutative algebras or Lie algebras) one can consider new types of algebras as well. The idea to model these types by their multilinear operations, and to compute with spaces of operations like monoids with a composition-multiplication, goes back to Lazard's analyzers [Laz55] and is now known as the theory of operads. The concept of Koszul duality, being well-known for quadratic algebras, exists also for operads (see [GK94]). As for the types $\mathcal{L}ie$ and $\mathcal{C}om$, the theory of algebras over one operad is strongly connected with the theory of algebras over the dual operad.

The first applications of operads, introduced by Boardman and Vogt and named by May ([BV], [May72]), were in algebraic topology. In the 1990's the theory of operads developed rapidly. Operads are important tools for deformation quantization and formality, and further influences came from mathematical physics, where operads of moduli spaces of Riemann surfaces with marked points are relevant for string field theories (cf.[KM01], [Kon]).

Strong homotopy algebras and other types of algebras with infinitely many different operations (like $A_\infty, B_\infty, C_\infty$) appear in the context of operads (cf. [GJ94]).

Operations modeled by an operad have multiple inputs but only one output. Thus they can be represented by rooted trees, where leaves correspond to inputs and the root corresponds to the output. If there are no relations between the operations (the case of free operads), we can compute in vector spaces of trees, and a natural grafting of trees corresponds to the operad-composition (cf. [MSS]).

The two operations $\prec, \succ$ of the dendriform operad $\mathcal{D}end$ define operations on the set of planar binary trees which provide the space of all planar binary trees with the structure of a free dendriform algebra (see [Lod01]). Loday and Ronco [LR98] introduced a natural Hopf algebra structure of the free dendriform algebra on one generator. The Loday-Ronco Hopf algebra is in fact isomorphic to the noncommutative planar Connes-Kreimer Hopf algebra and the Brouder-Frabetti Hopf algebra ([Hol03], [Foi]). Thus all three have the structure of a dendriform Hopf algebra.

In this work, we consider examples of "non-classical" Hopf algebras. Just as $\mathcal{P}$-algebras, i.e. algebras over different operads $\mathcal{P}$, make sense, $\mathcal{P}$-Hopf algebras are defined (see Chapter 3). Since we deal with one (coassociative) cooperation, there are coherence conditions which the operad $\mathcal{P}$ has to fulfill, such that the tensor product of $\mathcal{P}$-algebras is again a $\mathcal{P}$-algebra.

We do not include objects with various not necessarily associative operations and not necessarily coassociative cooperations, but the given definition is general enough to include dendriform Hopf algebras (compare [Lod03b]).

To avoid problems with antipodes, we restrict to the case of filtered or graded Hopf algebras.

Apart from dendriform Hopf algebras, we describe Hopf algebra structures over free operads. Therefore combinatorial operations and admissible labelings for various sorts of rooted trees are needed. We develop these tools in Chapter 2. As an application, we describe the Loday-Ronco dendriform Hopf algebra. We also describe its isomorphisms with the noncommutative planar Connes-Kreimer Hopf algebra and with a Hopf algebra of Brouder and Frabetti (in Chapter 3).

Let $\mathcal{M}ag$ be the operad freely generated by a non-commutative non-associative binary operation $\vee^2(x_1, x_2)$ also denoted by $x_1 \cdot x_2$. More exactly, the free operad $\mathcal{M}ag$ is the symmetrization of the free non-Σ operad generated by $\vee^2$.

Let similarly $\mathcal{M}ag_\omega$ be freely generated by operations $\vee^n$, one for each $2 \leq n \in \mathbb{N}$. A basis for the space of n-ary operations is given by reduced planar rooted trees with n leaves. (This is the operad of Stasheff polytopes, see [Sta97]).

The suboperad $\mathcal{M}ag$ contains only the binary trees. Thus free algebras over $\mathcal{M}ag$ have planar binary rooted trees (with labeled leaves) as a vector space basis. These free algebras are called magma algebras, they are free non-associative.

For any operad $\mathcal{P}$ that fulfills the coherence conditions noted above, the free $\mathcal{P}$-algebra generated by a set X of variables is always a $\mathcal{P}$-Hopf algebra with the diagonal or co-addition Δ_a as a comultiplication. The $\mathcal{P}$-algebra homomorphism Δ_a is determined by $\Delta_a(x) = x \otimes 1 + 1 \otimes x$, for all $x \in X$.

It is natural to pose the question of primitive elements. In the classical case, where the operad is the operad $\mathcal{A}s$ of associative algebras, Lie polynomials occur as the primitive elements. The theorem of Friedrichs (cf. [Reu]) characterizes Lie polynomials as the primitive elements of the free associative algebra equipped with the coaddition.

In a natural way, there exists an operad Prim$\mathcal{P}$, and Prim$\mathcal{A}s = \mathcal{L}ie$. In the commutative case, only homogeneous elements of degree 1 are primitive, i.e. Prim$\mathcal{C}om =$ Vect. A table of pairs $\mathcal{P}$, Prim$\mathcal{P}$ (triples in fact, if coassociativity of the comultiplication is replaced by something different) is given by Loday and Ronco in ([LR04]), and we are going to add some new cases to the list.

We look at free $\mathcal{M}ag$- and $\mathcal{M}ag_\omega$-algebras equipped with the co-addition, i.e. we are interested in the operads Prim$\mathcal{M}ag$ and Prim$\mathcal{M}ag_\omega$.

Although these algebras are free non-associative, results of Drensky-Gerritzen [DG03] and Gerritzen [Ger04a] on a canonical exponential function in non-associative variables show that they have a rich structure.

In such free $\mathcal{P}$-algebras, primitive elements for the co-addition are also constants for the partial derivatives with respect to the variables. Whereas the primitive elements do not form a $\mathcal{P}$-algebra, the constants do form a $\mathcal{P}$-algebra. For example, the algebra of constants of the free associative algebra $K\langle X\rangle$ is generated by Lie polynomials ([Fal]).

We consider Taylor expansion for polynomials in non-associative variables, generalizing classical Taylor expansion or Taylor expansion in associative algebras (cf. [Dre84, Dre85], and [Ger98, Ger03]). These Taylor expansions yield projectors on constants.

We examine the first generators and relations needed to describe Prim$\mathcal{M}ag$ and Prim$\mathcal{M}ag_\omega$. The first non-trivial relation is a non-associative Jacobi identity.

Reviewing a result of [GH03], also obtained by Shestakov and Umirbaev [SU02], we show that it does not suffice to use n-ary operations for $n \leq 3$ to generate the operads Prim$\mathcal{M}ag$ and Prim$\mathcal{M}ag_\omega$.

The same is true for the (operad of) primitive elements of dendriform algebras. The results of Ronco (see [Ron00a], [Ron02], [Lod03b]) show that Prim$\mathcal{D}end$ is the operad $\mathcal{B}race$ of brace algebras.

In the case of dendriform Hopf algebras, the question of primitive elements is the question of primitive elements in the Loday-Ronco Hopf algebra, or equivalently in the noncommutative Connes-Kreimer Hopf algebra.

In order to describe the operads Prim$\mathcal{M}ag_\omega$ and Prim$\mathcal{M}ag$, we describe the graded duals of the given co-addition $\mathcal{M}ag$-Hopf algebras. These duals are equipped with a commutative multiplication $\sqcup\!\sqcup$, which generalizes the shuffle multiplication known for free associative algebras. We show in Chapter 4, that these commutative algebras are freely generated by the primitive elements (and dually the given co-addition Hopf algebras are connected co-free). This is an analogon of the Poincaré-Birkhoff-Witt theorem.

We determine the generating series for the operads Prim$\mathcal{M}ag$ and Prim$\mathcal{M}ag_\omega$. We show that the dimension of Prim$\mathcal{M}ag(n)$ is related to the log-Catalan numbers $c'_1 = 1,\ c'_2 = 1,\ c'_3 = 4,\ c'_4 = 13,\ c'_5 = 46,\ c'_6 = 166,\ \ldots$ by

$$\dim \mathrm{Prim}\mathcal{M}ag(n) = (n-1)!c'_n.$$

By a recursive method, we show how the spaces Prim_n of homogeneous elements of degree n can be described as Σ_n-modules. As in the case of $\mathcal{L}ie_n$, this method only works for small n. For dendriform algebras, Ronco proved a Milnor-Moore type theorem (see [Ron00a, Ron02]) using Eulerian idempotents as projections onto Prim$\mathcal{D}end$. Similar projectors may be defined for $\mathcal{M}ag$ and $\mathcal{M}ag_\omega$.

In the last section of Chapter 4, we sketch a generalized Lazard-Lie theory (see [Laz55], [Hol01]) for $\mathcal{P}$-Hopf algebras.

Tuples of primitive elements occur as elements of the first cohomology group H^1. Elements of H^2 and H^3 are obstructions (for uniqueness and existence) that are relevant for a desired classification of (complete) $\mathcal{P}$-Hopf algebras.

In more detail, the content of this work can be outlined as follows.

In Sections 1.1-1.3 we recall the definitions, together with first examples, for Σ-vector spaces, operads and non-Σ operads, $\circ_i$-operations, generating series, algebras (and coalgebras) over operads, and especially free (free complete) algebras and cofree coalgebras.

In Section 2.1, we introduce the notions of admissibly labeled (planar or abstract) rooted trees, where a sequence of sets M_k contains the allowed labels for vertices of arity k, see Definition (2.1.6). This notion is useful to present operads (especially free operads), several examples of dendriform Hopf algebras and $\mathcal{M}ag$- or $\mathcal{M}ag_\omega$-Hopf algebras in a consistent way.

Among other integer sequences, we discuss the sequences of Catalan numbers and log-Catalan numbers, see Example (2.1.9), which will be relevant in Chapter 4.

We also recall some important operations on trees, like grafting and de-grafting operators.

In Section 2.2, the connection between trees and parenthesized words is explained. We describe cuts of trees, and we introduce leaf-restrictions, see Lemma (2.2.5), and leaf-splits. Shuffles for trees are also defined.

In Section 2.3, we first focus on planar binary trees, and a (right) comb representation for them, which is used for a bijection between such trees and forests of not necessarily binary planar trees, see Lemma (2.3.11). Then we consider Stasheff polytopes and super-Catalan numbers.

Section 2.4 is a continuation both of Sections 2.1-3 and Sections 1.1-3. We present free (non Σ-)operads using admissibly labeled trees, see Lemma (2.4.4). Especially we consider the operads $\mathcal{M}ag$ and $\mathcal{M}ag_\omega$ which are the main topic of Chapter 4.

After a brief introduction to classical Hopf algebra theory, given in Section 3.1, we consider in Sections 3.2 and 3.3 $\mathcal{P}$-Hopf algebras for operads $\mathcal{P}$ with a coherent unit action.

Coherent means that the tensor product of unitary $\mathcal{P}$-algebras A, B is equipped with the structure of a $\mathcal{P}$-algebra in a way such that elements $a \otimes 1$ (for $a \in A$) generate a $\mathcal{P}$-algebra isomorphic to A, and elements $1 \otimes b$ (for $b \in B$) generate a $\mathcal{P}$-algebra isomorphic to B. Operads like $\mathcal{C}om$, $\mathcal{A}s$, $\mathcal{M}ag$, $\mathcal{C}mg$, $\mathcal{P}ois$ and $\mathcal{D}end$ are equipped with a coherent unit action.

In the definition of a $\mathcal{P}$-Hopf algebra, see Definitions (3.3.1) and (3.3.4), the comultiplication Δ is required to be a coassociative map which is also a morphism of unitary $\mathcal{P}$-algebras.

Given a graded $\mathcal{P}$-Hopf algebra A, we also consider the graded dual A^{*g}, which is a graded $\mathcal{A}s$-algebra with respect to the operation Δ^*, see Lemma (3.3.10).

In Sections 3.4-3.6 we focus on $\mathcal{D}end$-Hopf algebras. The Hopf algebra structures on rooted trees introduced by Loday and Ronco, Brouder and Frabetti, and the

(non-commutative) Connes-Kreimer Hopf algebra are described. We review the isomorphisms between them in an elementary way, see Propositions (3.5.7) and (3.6.6).

In Section 3.7 we show that there exists an operad Prim$\mathcal{P}$, see Lemma (3.7.1), with Prim$\mathcal{P}(n)$ corresponding to the homogeneous multilinear primitive elements of degree n in the free $\mathcal{P}$-algebra (on n variables). We give the classical examples, and we recall the Milnor-Moore theorem.

We sketch results of Ronco and Loday-Ronco, which show that the operads of brace algebras and (non-dg) B_∞-algebras are of the form Prim$\mathcal{P}$.

In Section 4.1 we collect some information about the operads $\mathcal{M}ag$ and $\mathcal{M}ag_\omega$. Both are equipped with a (canonical) coherent unit action. We view the elements of the free $\mathcal{M}ag$-algebras $K\{X\}$ and $\mathcal{M}ag_\omega$-algebras $K\{X\}_\infty$ as polynomials, while monomials correspond to admissibly labeled planar trees.

We consider formal derivatives for $\mathcal{M}ag$- and $\mathcal{M}ag_\omega$-algebras in Section 4.2. We give a concrete description how to compute the partial derivatives ∂_k of tree monomials, using the concept of leaf-restriction (introduced in Section 2.2), see Proposition (4.2.6).

We show in Section 4.3 that the free unitary $\mathcal{M}ag_\omega$-algebra $K\{X\}_\infty$ together with the co-addition map Δ_a is a strictly graded $\mathcal{M}ag_\omega$-Hopf algebra, containing the free unitary $\mathcal{M}ag$-algebra $K\{X\}$ as a sub-Hopf algebra, see Proposition (4.3.3). We describe how the co-addition acts on monomials, see Proposition (4.3.5).

We introduce generalized differential operators ∂_T and prove properties of these operators, see Proposition (4.3.7).

In Section 4.4, see Proposition (4.4.3), we introduce the concept of Taylor expansions for $\mathcal{M}ag_\omega$-algebras $K\{X\}_\infty/I$, generalizing classical Taylor expansions.

While in Sections 4.1 to 4.4 we prove properties of (free) $\mathcal{M}ag_\omega$-algebras, which may immediately be translated into properties of (free) $\mathcal{M}ag$-algebras, we start in Section 4.5 to give a description of constant and primitive elements of $K\{X\}_\infty$ and $K\{X\}$ separately.

The concept of Taylor expansion is used to describe the spaces of primitive elements Prim$\mathcal{M}ag(n)$ for degrees $n \leq 3$, see Proposition (4.5.6). In Proposition (4.5.5) the spaces of primitive elements Prim$\mathcal{M}ag_\omega(n)$ for degrees $n \leq 3$ are described.

We also prove that Prim$\mathcal{M}ag$ and Prim$\mathcal{M}ag_\omega$ can not be generated by quadratic and ternary operations, see Corollary (4.5.8).

In Section 4.6, see Propositions (4.6.1) and (4.6.5), we explicitly describe the graded duals of the $\mathcal{M}ag_\omega$-Hopf algebra $(K\{X\}_\infty, \Delta_a)$ and of the $\mathcal{M}ag$-Hopf algebra $(K\{X\}, \Delta_a)$. Therefore we consider a shuffle multiplication which is a sum of the shuffles (introduced in Section 2.2).

For the operads $\mathcal{M}ag$ and $\mathcal{M}ag_\omega$, we prove an analogon of the Poincaré-Birkhoff-Witt theorem, see Theorem (4.7.5) and Theorem (4.7.6), in Section 4.7. Since the proof is the same for the $\mathcal{M}ag$- and the $\mathcal{M}ag_\omega$-case, we treat both cases together, see Proposition (4.7.1) and Lemma (4.7.3).

In Section 4.8 we have a closer look at the operad Prim$\mathcal{M}ag$. Using the explicit generating function of the operad $\mathcal{M}ag$, we obtain from Theorem (4.7.5) the Corollary (4.8.1), which states that $\dim \mathrm{Prim}\mathcal{M}ag(n)$ is given by (a multiple of) the n-th log-Catalan number, for all n.

We describe Prim$\mathcal{M}ag(4)$ by means of representation theory, see Proposition (4.8.3), and we demonstrate how to compute a basis of highest weight vectors. We finish with the analogues of Corollary (4.8.1) and Proposition (4.8.3) for the case of Prim$\mathcal{M}ag_\omega$.

The aim of the last section, Section 4.9, is to show that the question for primitive elements is only the first step in a generalized Lazard-Lie theory for $\mathcal{P}$-Hopf algebras. The goal is to obtain classification results for (filtered, or graded, or complete) $\mathcal{P}$-Hopf algebras.

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Chapter 1

Basic Concepts.

1.1 Words and permutations

Let K be a field of characteristic 0, and let Vect_K be the category of vector spaces over K. By X we usually denote a non-empty countable set of variables. If X is infinite, we usually consider $X = \{x_1, x_2, x_3, \ldots\}$. If the cardinality $\#X$ of X is finite, $\#X = m$, we use the variables $x_1, \ldots, x_m$. We also consider X as a basis of the vector space $V_X = KX = \mathrm{Span}_K(X)$.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^* = \{1, 2, \ldots\}$.

Definition 1.1.1. For $m \in \mathbb{N}^*$, we define the set $\underline{m} := \{1, \ldots, m\}$. We also define $\underline{0} := \emptyset$. For $m \in \mathbb{N}$, let Σ_m be the symmetric group, acting from the left on the set $\underline{m}$ via $\sigma.i = \sigma(i)$. As a Coxeter group, we generate Σ_m by $\{\tau_1, \tau_2, \ldots, \tau_{m-1}\}$, where τ_i is the transposition exchanging i and $i+1$.

Remark 1.1.2. To fix notation, we recall the free objects of the categories of abelian semigroups, (not necessarily abelian) semigroups, and magmas. We also consider the associated categories of unitary semigroups. Words occur as elements of these free objects.

Elements of the free abelian semi-group $W_{\mathrm{Com}}(X)$ over X are commutative words $x_{i_1}^{\nu_1} x_{i_2}^{\nu_2} \cdots x_{i_r}^{\nu_r}$, $i_1 < i_2 < \ldots < i_r$, $r \geq 1$, $\nu_i \in \mathbb{N}$. Adjoining a unit 1 (empty word) we get the free abelian semi-group $W^1_{\mathrm{Com}}(X)$ with unit.

The free semigroup over X is denoted by $W_{\mathrm{As}}(X)$, the free semigroup with unit by $W^1_{\mathrm{As}}(X)$. We denote the concatenation $W^1_{\mathrm{As}}(X) \times W^1_{\mathrm{As}}(X) \to W^1_{\mathrm{As}}(X), (v, w) \mapsto v.w$ by a lower dot. The elements of $W^1_{\mathrm{As}}(X)$ are words $w = w_1.w_2 \ldots w_r$, $w_i \in X$ for all i. Here $r \in \mathbb{N}$ is the length of w. Elements σ of Σ_m are also written in one-line notation, i.e. as words $\sigma(1).\sigma(2) \ldots \sigma(m)$. (The length of a permutation is the number of inversions $i < j$ with $\sigma(i) > \sigma(j)$, though.)

A magma is just a set M equipped with a binary operation (usually denoted by $\cdot : M \times M \to M$). The elements of the free magma $W_{\mathrm{Mag}}(X)$ over X are parenthesized words. We are going to identify these words with binary rooted trees, see Section 2.2.

For any r, we have a left action of Σ_r on the set $W_{r,\mathrm{As}}(X)$ of words $w = w_1.w_2 \dots w_r$ of length r by place permutation: for $i = 1, \dots, r$, we move the i-th letter to position $\sigma(i)$, thus $\sigma.(w_1.w_2 \dots w_r) = w_{\sigma^{-1}(1)}.w_{\sigma^{-1}(2)} \dots w_{\sigma^{-1}(r)}$.

Definition 1.1.3. Let $\sigma \in \Sigma_n$. Let $(m_1, \dots, m_n)$ be an ordered partition of m, i.e. any tuple $(m_1, \dots, m_n)$ with $m_i \in \mathbb{N}^*$ (all i) and $m = m_1 + \dots + m_n$. We denote by $(m_1', \dots, m_n')$ the ordered partition $(m_{\sigma^{-1}(1)}, \dots, m_{\sigma^{-1}(n)})$ of m.

The block permutation $\sigma_{(m_1,\dots,m_n)} \in \Sigma_m$ acts on the set $\underline{m}$ in the following way: For each $1 \leq i \leq n$ it maps the interval $\{j : m_1 + \dots + m_{i-1} < j \leq m_1 + \dots + m_i\}$ strictly monotonic onto the interval $\{j' : m_1' + \dots + m'_{\sigma(i)-1} < j' \leq m_1' + \dots + m'_{\sigma(i)}\}$. In other words, $\sigma_{(m_1,\dots,m_n)}$ acts on the intervals of length $m_1, \dots, m_n$ in the same way as σ acts on $\underline{n}$. If $n = 3, m = 9$, $(m_1, m_2, m_3) = (2, 4, 3)$ and σ is the transposition exchanging 3 and 1, then $\sigma_{(m_1,m_2,m_3)}$ maps $\{7, 8, 9\}$, i.e. the third interval with respect to (m_1, m_2, m_3), onto $\{1, 2, 3\}$. The interval $\{3, 4, 5, 6\}$ becomes the interval $\{4, 5, 6, 7\}$, and $\{1, 2\}$ is mapped onto $\{8, 9\}$.

For every $n \in \mathbb{N}, m_1, \dots, m_n \in \mathbb{N}$, $m := m_1 + \dots + m_n$, there are maps

$$\Sigma_n \times \Sigma_{m_1} \times \dots \times \Sigma_{m_n} \to \Sigma_m$$

defined by

$$(\sigma, \gamma_1, \dots, \gamma_n) \mapsto \sigma_{(m_1,\dots,m_n)} \circ (\gamma_1 \times \dots \times \gamma_n)$$

(with $\gamma_1 \times \dots \times \gamma_n$ acting on $\{1, \dots, m_1\} \times \dots \times \{m - m_n + 1, \dots, m\} \subset \underline{m}$.)

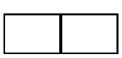
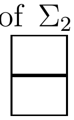
By K-linear extension, we get maps

$$\mu_{n;m_1,\dots,m_n} : K\Sigma_n \otimes K\Sigma_{m_1} \otimes \dots \otimes K\Sigma_{m_n} \to K\Sigma_m.$$

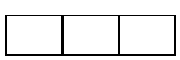
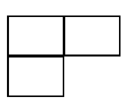
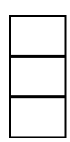
Remark 1.1.4. The maps $\mu_{n;m_1,\dots,m_n}$ turn the sequence $(K\Sigma_n)_{n \in \mathbb{N}}$ into a K-linear operad, see the next section.

Remark 1.1.5. The group ring $K\Sigma_n$ corresponds to the regular representation of Σ_n. We refer to [JK] for the representations of the symmetric groups.

The irreducible representations of Σ_n are symbolized by Young diagrams. For example the regular representation of Σ_2 is the sum of the trivial representation

 and the sign representation  .

In degree 3, the regular representation is given by

 $\oplus$ 2  $\oplus$  .

The symmetric group Σ_n is embedded into $GL_m(K)$, $m \geq n$, and there is a similar theory for GL-modules, cf. [Wey]. For any group G, a left (right) G-module is a vector space V over K together with a left (right) action of G.

1.2 Operads

The concept of operads and its precursors "analyseurs" and "compositeurs" (introduced by Lazard, see [Laz55]) is a good device to handle several types of algebras at once. From this point of view, the ability to form compositions is most important. On the one hand, looking at the corresponding free algebras, it is possible to insert elements into another. On the other hand, looking at the operations which characterize the algebra structure, we might combine operations to get new ones (possibly of a higher number of arguments).

An elegant way to introduce operads is to define an operad as a triple or monad ([Mac], chap. VI, cf. also [Fre98a]) in the monoidal category $(\mathrm{End}(\mathrm{Vect}_K), \circ)$, cf. [KM01]. One usually requires to have an action of the symmetric groups. The analogously defined object without the additional structure related to Σ_n-actions is called a non-Σ-operad.

Definition 1.2.1. Let Σ-Vect_K be the following category: Objects $\mathcal{V}$ are sequences $(\mathcal{V}(n))_{n\in\mathbb{N}}$ of vector spaces $\mathcal{V}(n)$ with a right Σ_n-action.
Morphisms are given by homomorphisms compatible with the Σ_n-action.

To every Σ-vector space $\mathcal{V}$ there is associated an endofunctor on Vect_K given by

$$F_{\mathcal{V}}(V) := \bigoplus_{n=0}^{\infty} \big(\mathcal{V}(n) \otimes V^{\otimes n}\big)_{\Sigma_n} = \bigoplus_{n=0}^{\infty} \mathcal{V}(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

Here Σ_n acts from the left on $V^{\otimes n}$ by place permutation, and $\big(\mathcal{V}(n)\otimes V^{\otimes n}\big)_{\Sigma_n}$ denotes the space of coinvariants for the diagonal action of Σ_n. (Thus $p.\sigma \otimes q_1 \otimes \ldots \otimes q_n$ and $p \otimes \sigma.(q_1 \otimes \ldots \otimes q_n)$ are identified.)

The category Σ-Vect_K can be identified with the full subcategory of the category $\mathrm{End}(\mathrm{Vect}_K)$ of functors $\mathrm{Vect}_K \to \mathrm{Vect}_K$ consisting of functors of the form $F_{\mathcal{V}}$. The composition $\circ$ of functors induces an associative bifunctor $\odot$ on $\mathrm{End}(\mathrm{Vect}_K)$ such that $(\mathrm{End}(\mathrm{Vect}_K), \odot)$ and $(\Sigma\text{-}\mathrm{Vect}_K, \odot)$ are monoidal categories in the sense of [Mac], chap. VII.

Unit object is the identity functor $\mathrm{Id}_{\mathrm{Vect}_K}$, which is given by the Σ-vector space $\mathcal{I}$ with $\mathcal{I}(1) = K, \mathcal{I}(n) = 0$ for $n \neq 1$.

Definition 1.2.2. Let a Σ-space $\mathcal{P}$, together with morphisms of functors

$$\mu : F_{\mathcal{P}} \odot F_{\mathcal{P}} \to F_{\mathcal{P}} \text{ and } 1 : \mathrm{Id} \to F_{\mathcal{P}}$$

be given such that the following diagrams are commutative:

$$\begin{tikzcd}
F_{\mathcal{P}} \odot F_{\mathcal{P}} \odot F_{\mathcal{P}} \arrow[r, "\mathrm{id} \odot \mu"] \arrow[d, "\mu \odot \mathrm{id}"'] & F_{\mathcal{P}} \odot F_{\mathcal{P}} \arrow[d, "\mu"] \\
F_{\mathcal{P}} \odot F_{\mathcal{P}} \arrow[r, "\mu"'] & F_{\mathcal{P}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Id} \odot F_{\mathcal{P}} \arrow[r, "1 \odot \mathrm{id}"] \arrow[dr, "="'] & F_{\mathcal{P}} \odot F_{\mathcal{P}} \arrow[d, "\mu"] & F_{\mathcal{P}} \odot \mathrm{Id} \arrow[l, "\mathrm{id} \odot 1"'] \arrow[dl, "="] \\
& F_{\mathcal{P}} &
\end{tikzcd}$$

Then $(\mathcal{P}, \mu, 1)$, or just $\mathcal{P}$, is called a (K-linear) operad. The morphism μ is called (operad-)composition.

Remark 1.2.3. The elements of $\mathcal{P}(n)$ are often viewed as abstract operations with n inputs and one output.

From the description of $\mathcal{P} \odot \mathcal{P}$ given by $F_{\mathcal{P}\odot\mathcal{P}} = F_{\mathcal{P}} \odot F_{\mathcal{P}}$ it follows that the composition $\mu = \Big(\mu_V : F_{\mathcal{P}}\big(F_{\mathcal{P}}(V)\big) \to F_{\mathcal{P}}(V)\Big)$ is explicitly given by maps

$$\mu_{n;m_1,\ldots,m_n} : \mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \ldots \otimes \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \ldots + m_n), \text{ all } n, m_1, \ldots, m_n$$

called composition maps.

The composition maps can be interpreted as follows: The map $\mu_{n;m_1,\ldots,m_n}$ combines n operations $q_1, \ldots, q_n$ of $m_i (i = 1, \ldots, n)$ arguments via an operation p of n arguments. The result, also denoted by $p(q_1, \ldots, q_n)$, is an operation of $m_1 + \ldots + m_n$ arguments.

The unit map $1 : \mathrm{Id} \to F_{\mathcal{P}}$, or equivalently $1 : K \to \mathcal{P}(1)$, is given by an element $\mathrm{id} \in \mathcal{P}(1)$ such that $\mu(\mathrm{id} \otimes p) = p = \mu(p \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id})$ for all $p \in \mathcal{P}$.

The associativity, unity and Σ-invariance conditions following from (1.2.2), like

$$\begin{aligned} &\mu_{n;m_1,\ldots,m_n}\big(p.\sigma \otimes q_1 \otimes \ldots \otimes q_n\big) = \\ &\qquad\qquad \mu_{n;m_{\sigma^{-1}1},\ldots,m_{\sigma^{-1}n}}\big(p \otimes q_{\sigma^{-1}1} \otimes \ldots \otimes q_{\sigma^{-1}n}\big).\sigma_{(m_1,\ldots,m_n)}, \\ &\qquad\qquad\qquad \text{all } p \in \mathcal{P}(n), q_i \in \mathcal{P}(m_i), \sigma \in \Sigma_n \end{aligned}$$

are the original axioms of the definition given by May [May72].

Morphisms of operads are morphisms of Σ-spaces which respect unit and composition maps.

Example 1.2.4. Given a vector space V, there is an operad $\mathcal{E}nd_V$ defined by

$$\mathcal{E}nd_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$$

with unit given by id_V and composition maps induced by the composition of maps. The right Σ_n-action is given by $p.\sigma = p \circ \tilde{\sigma}$, where $\tilde{\sigma}$ is the left Σ_n-action of place permutation on $V^{\otimes n}$ induced by σ.

There also is an operad $\mathcal{C}o\mathcal{E}nd_V$ defined by

$$\mathcal{C}o\mathcal{E}nd_V(n) = \mathrm{Hom}(V, V^{\otimes n}).$$

Now the right Σ_n-action is defined by composition with the right Σ_n-action on $V^{\otimes n}$ given by $(v_1 \otimes \ldots \otimes v_n).\sigma = \sigma^{-1}.(v_1 \otimes \ldots \otimes v_n)$.

Remark 1.2.5. For any operad $\mathcal{P}$, we can consider the given Σ-space just as a sequence of vector spaces and forget all Σ_n-actions. This leads to a "nonsymmetric" version of the concept of operads. It appeared earlier than the concept of (symmetric) operads, cf. the notion of a comp algebra in [Gers63].

Definition 1.2.6. A K-linear non-Σ operad (or nonsymmetric operad) $\underline{\mathcal{P}}$ is a sequence $\underline{\mathcal{P}}(n)_{n\in\mathbb{N}}$ of vector spaces together with maps

$$\mu_{n;m_1,\dots,m_n} : \underline{\mathcal{P}}(n) \otimes \underline{\mathcal{P}}(m_1) \otimes \ldots \otimes \underline{\mathcal{P}}(m_n) \to \underline{\mathcal{P}}(m_1 + \ldots + m_n), \text{ all } n, m_1, \ldots, m_n$$

and a unit map $1 : K \to \underline{\mathcal{P}}(1)$ fulfilling the associativity and unity conditions indicated above.

Remark 1.2.7. By convention, in order to notationally distinguish operads from non-Σ operads, we use underlining (cf.[MSS]) to indicate that a non-Σ operad is given.

Definition 1.2.8. Let $\mathcal{P}$ be a K-linear operad (or non-Σ operad), and let the composition maps be denoted by

$$\mu_{n;m_1,\dots,m_n} : \mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \ldots \otimes \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \ldots + m_n), \text{ all } n, m_1, \ldots, m_n.$$

Then there are defined $\circ_i$-operations

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(m + n - 1), \text{ all } n, m \geq 1, \ 1 \leq i \leq n$$

mapping $p \otimes q$ onto

$$\mu_{n;1,\dots,m,\dots,1}(p \otimes \overset{1}{\mathrm{id}} \otimes \ldots \otimes \overset{i}{q} \otimes \ldots \otimes \overset{n}{\mathrm{id}}).$$

Remark 1.2.9. Similar to the $\circ_i$-operations of Gerstenhaber (cf. [Gers63]), the $\circ_i$-operations for operads fulfill, given any iteration

$$(h \circ_i p) \circ_j q : \mathcal{P}(r) \otimes \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(m + n + r - 2),$$

the condition

$$(h \circ_i p) \circ_j q = \begin{cases} (h \circ_j q) \circ_{i+m-1} p & : 1 \leq j \leq i-1 \\ h \circ_i (p \circ_{j-i+1} q) & : i \leq j \leq i+n-1 \\ (h \circ_{j-n+1} q) \circ_i p & : i+n \leq j. \end{cases}$$

The composition maps are determined by these operations. To define a (non-Σ) operad it thus suffices to give $\circ_i$-operations fulfilling the condition above, and to specify an operad unit, see [MSS], p. 45.

Remark 1.2.10. Although we consider K-linear operads and non-Σ operads, it should be noted that non-Σ operads can more generally be defined for arbitrary monoidal categories different from $(\mathrm{Vect}_K, \otimes)$, and that operads can be defined for arbitrary symmetric monodial categories (cf. [MSS], II.1). The original definition of Boardman, Vogt, and May (see [May72]) made use of the category of topological spaces.

Remark 1.2.11. For every non-Σ operad $\underline{\mathcal{P}}$ there is an operad $\mathcal{P}$, called the symmetrization of $\underline{\mathcal{P}}$, with $\mathcal{P}(n) = \underline{\mathcal{P}}(n) \otimes_K K\Sigma_n$ (all n) and composition maps induced by the maps of $\underline{\mathcal{P}}$ and the maps $\mu_{n;m_1,\ldots,m_n} : K\Sigma_n \otimes K\Sigma_{m_1} \otimes \ldots \otimes K\Sigma_{m_n} \to K\Sigma_m$ (see Section 1.1).

We will also say that a given operad $\mathcal{P}$ is regular (often also called non-Σ in the literature), if $\mathcal{P}$ is of the form $\mathcal{P}(n) = \underline{\mathcal{P}}(n) \otimes_K K\Sigma_n$ (all n) for some non-Σ operad $\underline{\mathcal{P}}$.

Definition 1.2.12. The generating series of a non-Σ operad $\underline{\mathcal{P}}$ is

$$f^{\underline{\mathcal{P}}}(t) := \sum_{n\geq 1} \big(\dim \underline{\mathcal{P}}(n)\big) t^n$$

The generating series of an operad $\mathcal{P}$ is

$$f^{\mathcal{P}}(t) := \sum_{n\geq 1} \frac{\dim \mathcal{P}(n)}{n!} t^n$$

Example 1.2.13. The operad $\mathcal{A}s$ given by $\mathcal{A}s(n) = K\Sigma_n$, $n \geq 1$ is the symmetrization of the operad $\underline{\mathcal{A}s}$ given by $\underline{\mathcal{A}s}(n) = K$ (all $n \geq 1$). The generating series is

$$\sum_{n\geq 1} t^n = \frac{t}{1-t}$$

and it will become clear in the next section, that $\mathcal{A}s$ describes the operad of (non-unitary) associative algebras.

1.3 Algebras and coalgebras over operads

Definition 1.3.1. For any operad $\mathcal{P}$, a $\mathcal{P}$-algebra consists of a vector space A together with a morphism $\gamma = (\gamma(n)) : \mathcal{P} \to \mathcal{E}nd_A$ of operads.

Each $p \in \mathcal{P}(n)$ yields a multilinear operation of n arguments on A.

Equivalently, a $\mathcal{P}$-algebra is a vector space A together with a family of Σ_n-invariant morphisms $\gamma_A(n) : \mathcal{P}(n) \otimes A^{\otimes n} \to A$, called structure maps.

Morphisms of $\mathcal{P}$-algebras $A \to B$ are K-linear maps $\varphi : A \to B$ compatible with the corresponding structure maps $\gamma_A(n), \gamma_B(n)$, i.e. such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{P}(n) \otimes_{\Sigma_n} A^{\otimes n} \arrow[r, "\gamma_A(n)"] \arrow[d, "\mathrm{id} \otimes_{\Sigma_n} \varphi^{\otimes n}"'] & A \arrow[d, "\varphi"] \\
\mathcal{P}(n) \otimes_{\Sigma_n} B^{\otimes n} \arrow[r, "\gamma_B(n)"'] & B
\end{tikzcd}$$

The space $F_{\mathcal{P}}(V) = \bigoplus_{n=0}^{\infty} \mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$ is the (underlying space of the) free $\mathcal{P}$-algebra generated by the space V. Its structure maps are induced by the composition maps $\mu = \mu_{n;m_1,\ldots,m_n}$.

Example 1.3.2. The free commutative associative algebra generated by a vector space V is the symmetric algebra $\bigoplus_n (V^{\otimes n})_{S_n}$, where the sum starts with $n = 0$ to get the free unitary commutative algebra, and with $n = 1$ for the non-unitary case.

Thus, the operad $\mathcal{C}om$ whose algebras are commutative associative algebras (not necessarily unitary), is given by

$$\mathcal{C}om(n) = K$$

(trivial representation of Σ_n) for each $n \geq 1$, and $\mathcal{C}om(0) = 0$.

The generating series is

$$\sum_{n \geq 1} \frac{t^n}{n!} = \exp(t) - 1.$$

Choosing

$$\mathcal{A}s(n) = K\Sigma_n$$

the (module of the) regular representation for each $n \geq 1$, one gets the operad of associative (not necessarily unitary) algebras, with free algebra functor

$$F_{\mathcal{A}s}(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}.$$

Lie algebras and Poisson algebras yield classical examples of operads, too. The operad $\mathcal{L}ie$ is given by the $(n-1)!$-dimensional K-spaces generated by multilinear bracket monomials with respect to Jacobi identity and anti-symmetry, cf. [GK94]. For $\mathcal{P}ois$ one needs a commutative associative binary operation and a Lie bracket operation $[,]$. They are related by the identity $[a, b \cdot c] = b \cdot [a, c] + [a, b] \cdot c$. (See Section 2.4 for the construction of operads by generators and relations.)

Remark 1.3.3. We note that the operads $\mathcal{C}om$ and $\mathcal{A}s$ and also $\mathcal{P}ois$ and $\mathcal{M}ag$ (the operad of magma algebras) are defined in a way such that their algebras are not necessarily unitary.

For example, the elements of the free $\mathcal{C}om$-algebra $F_{\mathcal{C}om}(V_X)$, V_X the vector space with basis X, are polynomials in commutative variables from X without constant terms, $F_{\mathcal{C}om}(V_X) = \overline{K[X]}$ (called the augmentation ideal of the free unitary commutative algebra).

Definition 1.3.4. (cf.[Fre98a], 1.4.)

A $\mathcal{P}$-algebra A is called nilpotent, if for sufficiently large n,

$$\gamma_A(n)(\mu \otimes a_1 \otimes \dots a_n) = 0$$

for all $a_1, \dots, a_n \in A$, $\mu \in \mathcal{P}(n)$.

An ideal of a $\mathcal{P}$-algebra A is a subspace I of A, such that

$$\gamma_A(n)(\mu \otimes a_1 \otimes \dots \otimes a_{n-1} \otimes b) \in I$$

for all $\mu \in \mathcal{P}(n), a_i \in A, b \in I$.

Remark 1.3.5. Given an ideal I in a $\mathcal{P}$-algebra A, the quotient A/I is again a $\mathcal{P}$-algebra.

If A is a $\mathcal{P}$-algebra together with a sequence $I_n, n \geq 1$ of ideals, such that the $\mathcal{P}$-algebras A/I_n are nilpotent, one can construct the completion $\widehat{A} = \varprojlim A/I_n$ of A with respect to the topology given by (I_n).

Definition 1.3.6. A complete $\mathcal{P}$-algebra is a $\mathcal{P}$-algebra A together with a sequence $I_n, n \geq 1$ of ideals, such that A/I_n is nilpotent for all n and such that $A = \varprojlim A/I_n$.

The free complete $\mathcal{P}$-algebra generated by a vector space V is given by

$$\widehat{F}_{\mathcal{P}}(V) := \prod_{n=1}^{\infty} \mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

If $V = V_X,\ X = \{x_1, \ldots, x_m\}$, the elements of $\widehat{F}_{\mathcal{P}}(x_1, \ldots, x_m)$ are called $\mathcal{P}$-power series in variables $x_1, \ldots, x_m$.

Example 1.3.7. For example, $\mathcal{C}om$-power series in variables $x_1, \ldots, x_m$ are power series (without constant terms) in commuting variables, $\widehat{F}_{\mathcal{C}om}(V_X) = \overline{K[[X]]}$.

Similarly, $\mathcal{A}s$-power series in variables $x_1, \ldots, x_m$ are power series (without constant terms) in non-commuting variables.

Remark 1.3.8. Dual to the notion of an operad, there is the notion of a co-operad $\mathcal{Q} = (\mathcal{Q}(n))_{n \in \mathbb{N}}$. The definition is analogous to the definition of operads. Now the endofunctor on Vect_K given by

$$\bigoplus_{n=0}^{\infty} \big(\mathcal{Q}(n) \otimes V^{\otimes n}\big)^{\Sigma_n}$$

has to be a comonad. (The space of invariants is used.)

We get (co-)composition maps $\mathcal{Q}(i_1 + \ldots + i_n) \to \mathcal{Q}(n) \otimes \mathcal{Q}(i_1) \otimes \ldots \otimes \mathcal{Q}(i_n)$.

If $\mathcal{P}^* = (\mathcal{P}(n))^*_{n \in \mathbb{N}}$ denotes the K-linear dual of a Σ-space $\mathcal{P}$, and if we assume all $\mathcal{P}(n)$ to be finite dimensional, the axioms for $\mathcal{P}$ being an operad correspond to the axioms for $\mathcal{P}^*$ being a co-operad.

Definition 1.3.9. (cf.[MSS], p.165)

Let $\mathcal{P}$ be an operad. We assume that all $\mathcal{P}(n)$ are finite dimensional.

A $\mathcal{P}$-coalgebra consists of a vector space C together with a morphism $\lambda_C = (\lambda_C(n)) : \mathcal{P} \to \mathcal{C}o\mathcal{E}nd_C$ of operads.

Equivalently, a $\mathcal{P}$-coalgebra is a vector space C together with a family of Σ_n-invariant morphisms $\lambda_C(n) : \mathcal{P}(n) \otimes C \to C^{\otimes n}$, called structure maps.

Morphisms of $\mathcal{P}$-coalgebras $C \to D$ are K-linear maps $\varphi : C \to D$ compatible with the corresponding structure maps.

A $\mathcal{P}$-coalgebra C is called connected (see [Fre97], §4) or (co-)nilpotent (see [MSS], p.165), if the following condition holds:

$$\text{for all } c \in C \text{ there is } N \in \mathbb{N} \text{ such that for } n > N, \lambda_C(n)(\mu \otimes c) = 0, \text{ all } \mu \in \mathcal{P}(n).$$

Remark 1.3.10. The space $F^c_{\mathcal{P}}(V) = \bigoplus_{n=0}^{\infty}\big(\mathcal{P}^*(n)\otimes V^{\otimes n}\big)^{\Sigma_n}$ is the underlying space of the cofree (co-)nilpotent $\mathcal{P}$-coalgebra (co-)generated by the space V. Its structure maps are induced by the dual composition maps

$$\mathcal{P}^*(i_1+\ldots+i_n) \to \mathcal{P}^*(n)\otimes\mathcal{P}(i_1)^*\otimes\ldots\otimes\mathcal{P}^*(i_n)$$

of (1.3.8).

If the characteristic of K is 0, for any vector space V over K there is a projection $\pi : V \to V^{\Sigma_n}$, $v \mapsto \frac{1}{n!}\sum_{\sigma\in\Sigma_n}\sigma(v)$ onto the space of invariants.

Hence $V_{\Sigma_n} \cong V/\ker\pi \cong V^{\Sigma_n}$. Thus via (non-canonical) vector space isomorphisms between $\mathcal{P}^*(n)$ and $\mathcal{P}(n)$ it is possible to provide the free $\mathcal{P}$-algebra $F_{\mathcal{P}}(V) = \bigoplus_{n=0}^{\infty}\mathcal{P}(n)\otimes_{\Sigma_n}V^{\otimes n}$ with the structure of a cofree (co-)nilpotent coalgebra.

The construction of non-nilpotent cofree coalgebras is more complicated (see [Fox93]).

Example 1.3.11. Let $K = \mathbb{Q}$. For $\mathcal{P} = \mathcal{A}s$, and V_X vector space with basis X, the words $w_1.w_2\ldots w_r$, $r \geq 1$, $w_i \in X$ form a basis of $F_{\mathcal{A}s}(V_X)$. Now the (co-)nilpotent cofree coalgebra-structure is given by deconcatenation, that is, for $\mu \in \mathcal{A}s(n), w = w_1\ldots w_r$, the image of $\mu\otimes w$ under $\lambda(n)$ is

$$\sum_{1\leq j_1<\ldots<j_{n-1}<r}(w_1\ldots w_{j_1})\otimes\ldots\otimes(w_{j_{n-1}+1}\ldots w_r).$$

Since all operations of $\mathcal{A}s$ can be built up from one binary multiplication $\tilde{\mu} \in \mathcal{A}s(2)$ (together with $\tilde{\mu}.\tau_1$, where τ_1 is the transposition), the coalgebra-structure is determined by the images of $\tilde{\mu}\otimes w$, given by

$$\Delta(w) := \sum_{j=1}^{r-1}(w_1\ldots w_j)\otimes(w_{j+1}\ldots w_r).$$

It is possible to extend the coalgebra structure onto $K\langle X\rangle = K1\oplus F_{\mathcal{A}s}(V_X)$, where 1 is the empty word, by setting

$$\Delta(w) := \sum_{j=0}^{r}(w_1\ldots w_j)\otimes(w_{j+1}\ldots w_r).$$

Usually (the vector space) $K\langle X\rangle$ together with Δ is called the standard tensor coalgebra.

Example 1.3.12. For $\mathcal{P} = \mathcal{C}om$, the cofree (co-)nilpotent coalgebra occurs as a subcoalgebra of the standard tensor coalgebra, namely the subcoalgebra of symmetric tensors.

Chapter 2

Some combinatorics of trees

2.1 Abstract and planar trees

For computations in not necessarily associative algebras, and also in operad theory, trees are very useful to symbolize the ways of associating variables (or arguments of an operation).

We have to make a difference between several types of trees. First we recall the notions of rooted trees and planar rooted trees. We skip the definition of graphs. A naive notion of a graph will suffice. For a more sophisticated notion, involving half-edges, see [MSS], §5.3.

Definition 2.1.1. A finite connected graph $\emptyset \neq T = (\mathrm{Ve}(T), \mathrm{Ed}(T))$, with a distinguished vertex ρ_T, is called an abstract rooted tree, if for every vertex $\lambda \in \mathrm{Ve}(T)$ there is exactly one path connecting λ and ρ_T.

The vertex ρ_T is called the root of T. Thinking of the edges as oriented towards the root, at each vertex there are incoming edges and one outgoing edge. Modifying the standard convention (but cf. also [MSS], p.50), we add to the root an outgoing edge that is not connected to any further vertex. (If we want to exclude this edge, we speak of the other edges as inner edges.)

We denote by ATree the set of abstract rooted trees.

Remark 2.1.2. In the literature, abstract rooted trees are often only called rooted trees. Since we are also going to deal with planar rooted trees, and since planar rooted trees are not special (abstract) rooted trees, we stress the word 'abstract'. We may skip the word 'rooted', because we are only going to consider rooted trees.

Definition 2.1.3. The height of a vertex $\lambda \in \mathrm{Ve}(T)$ is the number of edges separating it from ρ_T. The height of a rooted tree T is the maximum height of its vertices.

At a given vertex λ, the number n of incoming edges is called the arity ar_λ of λ. We write the set $\mathrm{Ve}(T)$ of vertices as a disjoint union $\bigcup_{n \in \mathbb{N}} \mathrm{Ve}^n(T)$. The vertices of arity 0 are called leaves, and we denote $\mathrm{Ve}^0(T)$ by $\mathrm{Le}(T)$. The elements of $\mathrm{Ve}^*(T) = \mathrm{Ve}(T) - \mathrm{Le}(T)$ are called internal vertices of T.

A tree T is called binary, if $\mathrm{Ve}^*(T) = \mathrm{Ve}^2(T)$, i.e. if $\mathrm{ar}_\lambda = 2$ for all internal vertices λ.

An abstract rooted tree T together with a chosen order of incoming edges at each vertex is called a planar rooted tree (or ordered rooted tree), see Example (2.1.5). We denote by PTree the set of planar rooted trees.

Remark 2.1.4. It is well-known that the number of planar rooted trees with n vertices is the n-th Catalan number

$$c_n = \frac{(2(n-1))!}{n!(n-1)!} = \sum_{l=1}^{n-1} c_l c_{n-l}.$$

The sequence of Catalan numbers is

$$c_1 = 1, c_2 = 1, c_3 = 2, c_4 = 5, c_5 = 14, c_6 = 42, c_7 = 132, c_8 = 429, c_9 = 1430, \ldots$$

with generating series $f(t) = \sum_{n=1}^{\infty} c_n t^n$ given by

$$\frac{1-\sqrt{1-4t}}{2}.$$

The numbers c_n also count the number of planar binary rooted trees with n leaves (or $2n-1$ vertices).

The numbers a_n of abstract rooted trees with n vertices, for $n = 1, \ldots, 11$, are

$$1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842.$$

The generating series $f(t) = \sum_{n=1}^{\infty} a_n t^n$ fulfills the equation

$$f(t) = \frac{t}{\prod_{n\geq 1}(1-t^n)^{a_n}}$$

or equivalently the equation

$$f(t) = t \exp\Big(\sum_{k\geq 1} \frac{f(t^k)}{k}\Big)$$

(cf. [Har]).

The numbers of abstract binary rooted trees with n leaves (or $2n-1$ vertices), for $n = 1, \ldots, 11$, are

$$1, 1, 1, 2, 3, 6, 11, 23, 46, 98, 207, 451$$

with generating series $f(t)$ given by (cf. [Pet03]) the equation

$$f(t) = t + \frac{1}{2} f(t)^2 + \frac{1}{2} f(t^2).$$

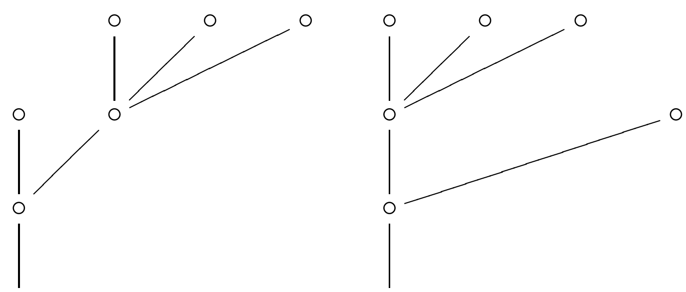
The series can thus be written in the form

$$\begin{aligned} f(t) &= 1 - \sqrt{1 - f(t^2) - 2t} \\ &= 1 - \sqrt{\sqrt{1 - f(t^4) - 2t^2} - 2t} = \dots \end{aligned}$$

Information on these integer sequences can be found in the On-Line Encyclopedia of Integer Sequences [Slo].

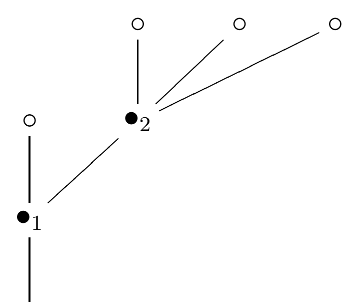
Example 2.1.5. We call rooted trees just trees, for short. We draw the root at the bottom (or bottom left, more exactly). For every vertex of a planar tree, the chosen order of incoming edges corresponds to an ordering of edges from left to right. Every drawing of a tree provides us with a planar structure, which we have to forget when dealing with abstract trees.

The drawings



represent the same abstract tree T (of height 2), but different planar trees T^1, T^2.

In our drawing, we can put labels at the vertices:



Here we have used different sets of labels for leaves and internal vertices.

The following definition of labeled trees and admissibly labeled trees is useful to include labeled trees in operad theory as well as other types of labelings.

Definition 2.1.6. Let M be a set and T a planar (or abstract) tree.

Then a labeling of T is a map $\nu : \mathrm{Ve}(T) \to M$. The tree T together with such a labeling is called a labeled tree.

Let a collection $M_0, M_1, M_2, \dots$ of sets be given, and let $M = \bigcup_{k \in \mathbb{N}} M_k$.

A labeling $\nu : \mathrm{Ve}(T) \to M$ of a planar (or abstract) tree T is called admissible, if the restrictions $\nu | \mathrm{Ve^k}(T)$ are maps $\mathrm{Ve^k}(T) \to M_k$, i.e. it holds that:

$$\nu(\lambda) \in M_k \text{ if } \ \mathrm{ar}_\lambda = k.$$

The set of planar rooted trees $T \in \mathrm{PTree}$ with admissible labeling from $(M_k)_{k\in\mathbb{N}}$ is denoted by $\mathrm{PTree}\{(M_k)_{k\in\mathbb{N}}\}$.

Given a fixed tree $T \in \mathrm{PTree}$, we define

$$T\{(M_k)_{k\in\mathbb{N}}\} := \{(T,\nu) : \ \nu \text{ admissible } \} \subseteq \mathrm{PTree}\{(M_k)_{k\in\mathbb{N}}\}.$$

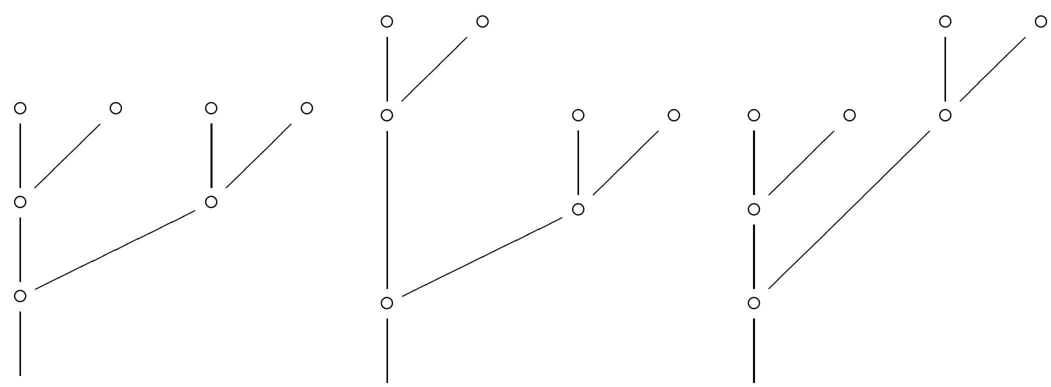
We make the same definitions for abstract rooted trees with admissible labelings. The set of these trees is denoted by $\mathrm{ATree}\{(M_k)_{k\in\mathbb{N}}\} = \bigcup_{T\in\mathrm{ATree}} T\{(M_k)_{k\in\mathbb{N}}\}$.

Remark 2.1.7. We can identify the set PTree with the set $\mathrm{PTree}\{M_k = \{\circ\}, k \in \mathbb{N}\}$, i.e. we consider non-labeled trees as trivially labeled trees.

In our drawings, we will often use labels $\bullet$ and $\circ$ to distinguish between vertices that count and vertices that do not count for a degree function.

More generally, one can work with weighted labels. Apart from the $\circ$-label (weight 0), we are only going to use labels of weight 1. The degree of a tree is then the number of all vertices that are not labeled by a $\circ$.

We mention that one can also associate levels to all vertices of a given tree, to distinguish for example between the trees



We are going to mention this type of trees only one time (in Section 2.4).

Definition 2.1.8. Let $a_n, n \geq 1$, be a sequence of integers with generating series $f(t) = \sum_{n=1}^{\infty} a_n t^n$.

The logarithmic derivative of $f(t)$ is the series

$$g(t) := \frac{\partial}{\partial t} \log\big(1 + f(t)\big),$$

and we say that the sequence $a'_n, n \geq 1$, with $\sum_{n=1}^{\infty} a'_n t^n = t \cdot g(t)$ is obtained from $a_n, n \geq 1$, by logarithmic derivation.

Example 2.1.9. The sequence of log-Catalan numbers c'_n, starting with

$$1, 1, 4, 13, 46, 166, 610, 2269, 8518, 32206, \ldots$$

has the generating series

$$\frac{2t}{3\sqrt{1-4t}-1+4t}\ .$$

Since

$$\frac{2}{3\sqrt{1-4t}-1+4t}=\frac{\partial}{\partial t}\log\Big(\frac{3-\sqrt{1-4t}}{2}\Big),$$

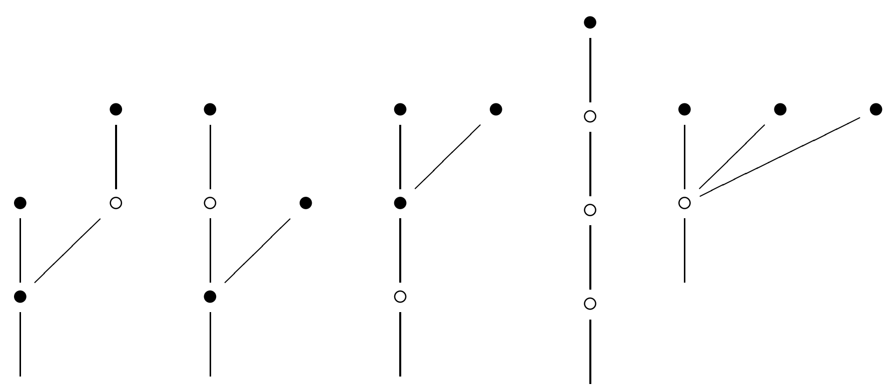
it is obtained by logarithmic derivation from the Catalan numbers $c_n, n \geq 1$.

In the set of all planar (rooted) trees with n vertices, the number of vertices with even arity is given by c'_n. The corresponding numbers of vertices with odd arity have the generating series

$$\begin{aligned}\sum_{n=1}^{\infty} nc_n t^n - \sum_{n=1}^{\infty} c'_n t^n &= \frac{t}{\sqrt{1-4t}} - \frac{2t}{(3-\sqrt{1-4t})\sqrt{1-4t}} \\ &= \frac{t(1-\sqrt{1-4t})}{(3-\sqrt{1-4t})\sqrt{1-4t}} = t^2+2t^3+7t^4+24t^5+86t^6+314t^7+\dots\end{aligned}$$

(see [Slo] A026641, [DS01] p. 258).

For example, in the set



of planar trees with $n = 4$ vertices, we count seven vertices with odd arity and $c'_4 = 13$ vertices with even arity.

Definition 2.1.10. Let T^1, T^2 be planar (or abstract) trees and let b be a leaf of T^1. Then the substitution of T^2 in T^1 at b, denoted by $T^1 \circ_b T^2$, is obtained by replacing the leaf b of T^1 by the root of T^2.

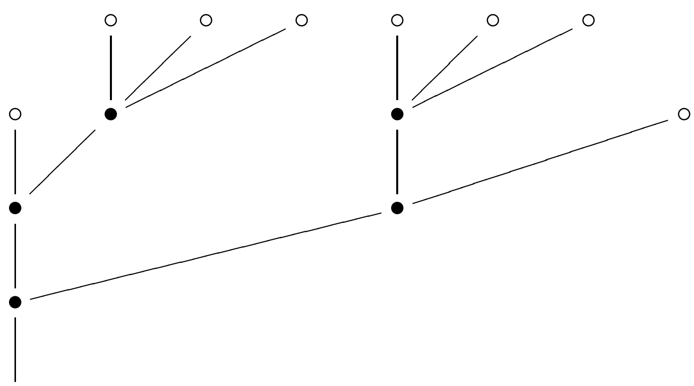
A word $T^1.T^2 \dots T^r$ (or an ordered tuple $(T^1, T^2, \dots T^r)$, not necessarily non-empty) of planar trees is called a planar forest. A disjoint union (or unordered tuple) of abstract trees is called an abstract forest.

In both cases, given a forest $T^1 \dots T^n$ of $n \geq 0$ trees, together with a label $\rho \in M$, there is a tree $T = \vee_\rho(T^1 \dots T^n)$ defined by introducing a new root of arity n and grafting the trees $T^1, \dots, T^n$ onto this new root. The new root gets the label ρ. In the planar case, the specified order determines the order of incoming edges at ρ_T. The tree T is called the grafting of $T^1 \dots T^n$ over ρ.

If there is no choice for a label ρ (i.e. there is only one label available) we simply write $\vee(T^1 \dots T^n)$. In the literature, $\vee(T^1 \dots T^n)$ is often denoted by $B_+(T^1 \dots T^n)$, e.g. in [Kre99].

Definition 2.1.11. A (planar or abstract) tree T is called reduced, if $\mathrm{ar}_\lambda \neq 1$ for all $\lambda \in \mathrm{Ve}(T)$. The set PRTree of planar reduced trees is identified with the set $\mathrm{PTree}\{(M_k)_{k\in\mathbb{N}}\}$ given by $M_1 = \emptyset$, and M_k (for $k = 0$ or $k \geq 2$) a one element set, $\{\circ\}$ say.

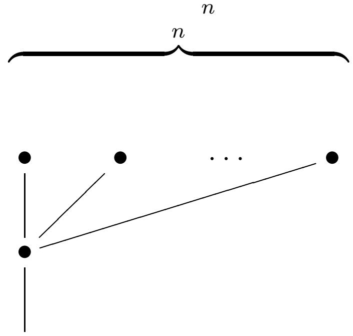
Example 2.1.12. The grafting $\vee(T^1, T^2)$ of T^1, T^2 from (2.1.5) is just



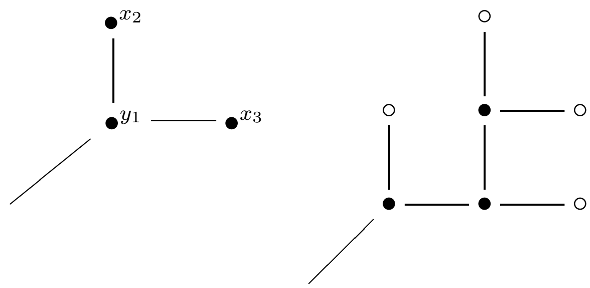
It is a reduced tree.

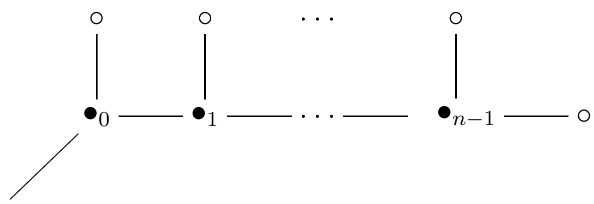
The following tree is not reduced:

The tree $\vee(\underbrace{\bullet \dots \bullet}_{n})$ is called n-corolla:



Examples for binary trees can be drawn as follows:

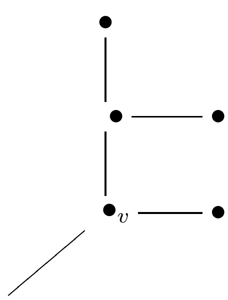




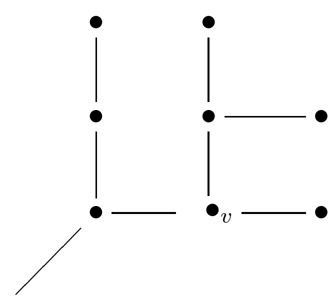
If no labels are given, the last binary tree in the picture above is called right comb (of height n). Left combs are defined analogously.

Definition 2.1.13. Let T be a planar (or abstract) tree, and let $v \in \mathrm{Ve}(T)$ be a vertex. The vertex v determines a subgraph S of T, called the full subtree of T with root v, such that: $v \in \mathrm{Ve}(S)$, and for every vertex $w \in \mathrm{Ve}(S)$ all incoming edges (vertices included) of w in T belong also to S.

Example 2.1.14.
The tree



is a full subtree of the tree



Remark 2.1.15. Every (abstract or planar) tree T is the grafting of a forest $\neg T$ uniquely determined by T. If the root ρ of T has arity n, the forest $\neg T$ consists of the n subtrees of T given by the n vertices connected to ρ by one edge. Especially, all trees in the forest $\neg T$ have heights less than the height of T.

If T is a planar binary tree with at least two leaves, then T can be uniquely written as the grafting $T = T^l \vee_\rho T^r := \vee_\rho(T^l, T^r)$ of its left tree T^l and its right tree T^r.

It is sometimes useful to call the empty set $\emptyset$ a tree (and to define its height to be -1); obviously $\emptyset$ cannot be written as a grafting. Let $\mathrm{PRTree}' := \mathrm{PRTree} \cup \{\emptyset\}$.

There is a canonical de-grafting map $\neg$ from labeled non-empty trees to forests of labeled trees (given by deleting the root together with its label). The operator $\neg$ is often denoted by B_-.

Given a planar tree T, there is a unique tree $\bar{T}$, recursively defined

$$\overline{\vee(T^1 \dots T^n)} = \vee(\bar{T}^n \dots \bar{T}^1),$$

where $\overline{\vee(\emptyset)} = \vee(\emptyset)$ and $\bar{\emptyset} = \emptyset$. In other words, $\bar{T}$ is obtained by mirroring T along the root axis.

It holds that $\overline{(\bar{T})} = T$. The trees T^1 and T^2 from (2.1.5) are in correspondence via $T \mapsto \bar{T}$.

2.2 Strings, reductions, and cuts

There is a correspondence between planar rooted trees and (irreducible) parenthesized strings. We sketch this correspondence in the general setting of admissibly labeled planar trees, where $M_0 = \{x_1, x_2, \dots\}$, $M_k = \{\mu_1, \mu_2, \dots\}(k \geq 1)$.

Kreimer's definition of irreducible parenthesized words in [Kre98] is completely analogous, but for the special case where $x_i = \mu_i$ (all i).

Remark 2.2.1. Given an admissibly labeled planar tree T, we recursively construct the corresponding parenthesized string.

If T consists of its root ρ_T, then ρ_T is a leaf labeled by some x_{i_1}. The corresponding parenthesized string is (x_{i_1}), i.e. an opening bracket followed by the letter x_{i_1} followed by a closing bracket.

Else, let ρ_T be labeled by μ_{i_1}, and let $T^1 \dots T^n$ be the forest of labeled trees which remains after removing the root with its incoming edges. Assume that T_j has got the corresponding parenthesized string w_j, all j. Then $(\mu_{i_1} w_1 \dots w_n)$ is the parenthesized string associated to T.

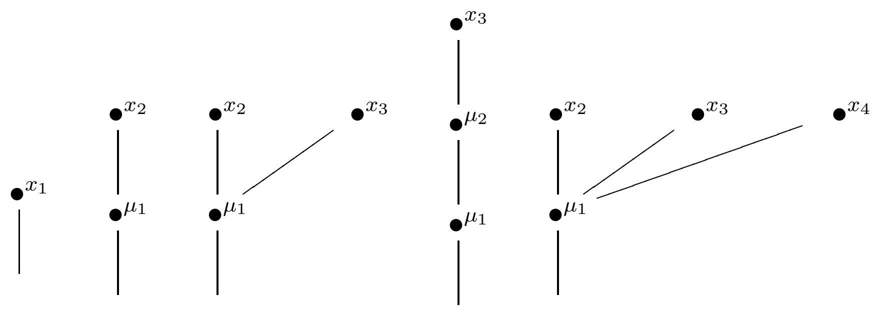
We get a string of letters and balanced brackets such that the leftmost opening bracket is matched by the rightmost closing bracket (irreducibility) and such that each letter has exactly one opening bracket on its lefthandside.

It is easy to see that the tree can be reconstructed from its string.

Reducible words are defined by concatenation of irreducible ones, thus they correspond to forests.

For abstract trees, there is a completely similar construction. The only difference is that some words have to be identified due to the missing order of incoming edges.

Example 2.2.2. The parenthesized strings $(x_1), (\mu_1(x_2)), (\mu_1(x_2)(x_3)), (\mu_1(\mu_2(x_3)))$, and $(\mu_1(x_2)(x_3)(x_4))$ represent the trees

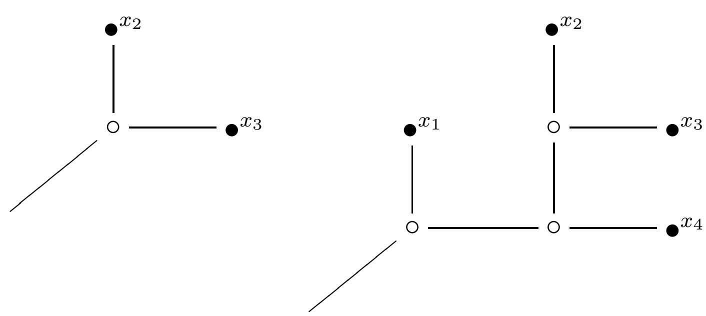


An empty pair of brackets without label is also allowed. It represents the empty tree.

Example 2.2.3. In the following we consider binary trees. Let $M_0 = \{x_1, x_2, \ldots\}$, and let M_2 be a one element set.

For every pair of brackets, the position of the closing bracket is forced once the position of the opening bracket is given. Thus one can omit the brackets and just use a letter c to mark an opening bracket given by an internal vertex.

For example the binary trees



can be represented by the strings cx_2x_3, $cx_1c^2x_2x_3x_4$.

Remark 2.2.4. Since the free magma generated by a set of variables X consists of parenthesized strings given by planar binary trees, we can call the set of planar binary trees with leaves labeled by X the free magma generated by X.

If a field K is given, we can pass from the free magma generated by X to the free magma algebra (similarly to passing from semi-groups or groups to semi-group algebras or group-algebras).

The representation given in Example (2.2.3) is the Malcev representation of the free magma algebra over $X = \{x_1, x_2, \ldots\}$ in the free associative algebra generated by $\{c, x_1, x_2, \ldots\}$. The free magma multiplication $\cdot$ corresponds to the operation $(v, w) \mapsto cvw$ in the free associative algebra.

Lemma 2.2.5. *Let $M_0 = \{x_1, x_2, \ldots\}, M_k = \{\mu_1, \mu_2, \ldots\}$ (for $k \geq 1$), and let T be a (planar or abstract) admissibly labeled tree, with corresponding parenthesized string w. Let $I \subseteq \mathrm{Le}(T)$ be a subset of the set of leaves.*

We can delete all pairs of brackets (together with its letter) that have no pair of bracket corresponding to a vertex from I in between, to obtain a parenthesized string $w|I$.

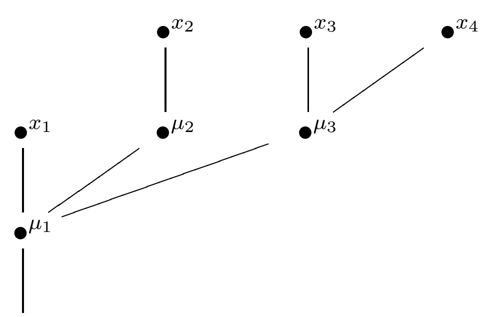
The string $w|I$ corresponds to a (not necessarily non-empty) admissibly labeled tree $T|I$. The set of vertices of $T|I$ corresponds to a subset of $\mathrm{Ve}(T)$, and the arity of v in $\mathrm{Ve}(T|I)$ is less or equal than the arity of the corresponding vertex in T.

Proof. The arity of a vertex v in T corresponds to the number of irreducible strings in between the pair of brackets corresponding to v. Removing strings as indicated does not increase this number, and we get an admissibly labeled tree $T|I$.

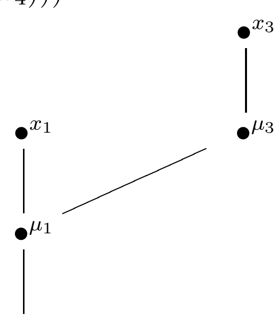
□

Definition 2.2.6. The tree $T|I$ is called the (non-reduced) leaf-restriction of T on $I \subseteq \mathrm{Le}(T)$.

Example 2.2.7. Consider the tree



with parenthesized string $(\mu_1(x_1)(\mu_2(x_2))(\mu_3(x_3)(x_4)))$.
The leaf-restriction on the first and third leaf is



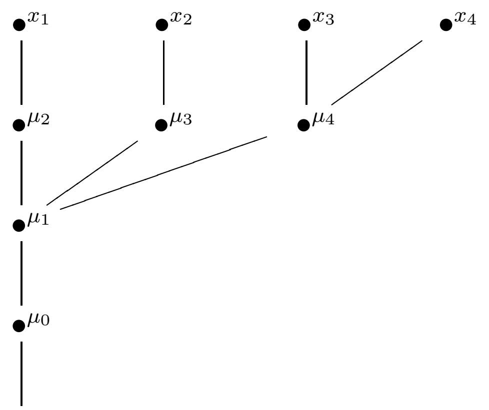
with parenthesized string $(\mu_1(x_1)(\mu_3(x_3)))$.

Remark 2.2.8. Leaf-restriction is an example for a process induced by the removal of some vertices of a tree. A similar process is induced by removing all vertices of arity 1: There is a (canonical) map red : PTree $\to$ PRTree (and a similar map for abstract trees), leaving the tree structure intact as much as possible. Admissible labelings (of all vertices of arity $\neq 1$) are preserved.

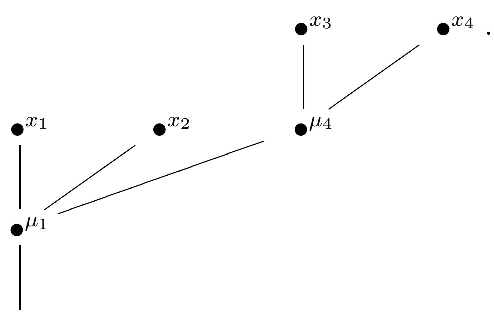
Definition 2.2.9. For T a (planar or abstract) tree, the tree $\mathrm{red}(T)$ is called the reduction of T. Its set of vertices is $\mathrm{Ve}(T) - \mathrm{Ve}^1(T)$, and for any pair v, v' of vertices

of red(T) there is an oriented path (or, equivalently, a path not passing the root) from v to v' in red(T) if and only there is such a path in T (cf. also [Ger04b])

Example 2.2.10. Consider the tree



Its reduction is



Definition 2.2.11. Let T be a (planar or abstract) tree with root ρ, and $C \subseteq \mathrm{Ve}(T)$. We call C an admissible cut of T, if for every vertex $v \in C$ all vertices of the full subtree given by v are also in C. The case $C = \emptyset$ is called the empty cut. The case $C = \mathrm{Ve}(T)$ is called the full cut.

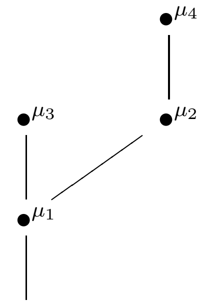
Given such an admissible cut, let $R^C(T)$ be the not necessarily non-empty tree (with root ρ, if $R^C(T) \neq \emptyset$), obtained by removing all vertices of C (together with their outgoing edges).

From T we can remove (the subgraph) $R^C(T)$ to get a (planar or abstract) forest $C(T)$ with set of vertices C.

The pair $(C(T), R^C(T))$ is called result of the cut C.

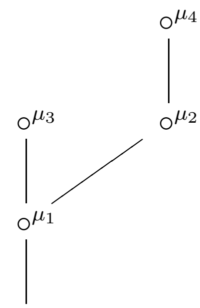
Remark 2.2.12. An admissible cut of T can also be defined as a non-empty subset of the set of (inner) edges of T such that for every vertex $v \in \mathrm{Ve}(T)$ on the path to the root there is at most one edge selected, cf. [CK98]. This definition leads to the same pair $(C(T), R^C(T))$ and is in fact equivalent, once we add the full and empty cut.

Example 2.2.13. Let T be the following planar tree:

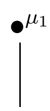


We are going to indicate (by $\circ$) which vertices are selected. Some admissible cuts of T are:

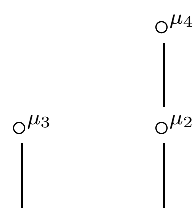
We get $R^C(T) = \emptyset$ and $C(T) = T$ as the result of the cut



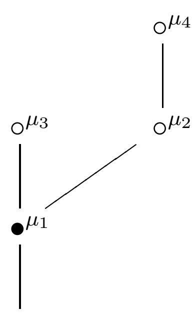
and $R^C(T) =$



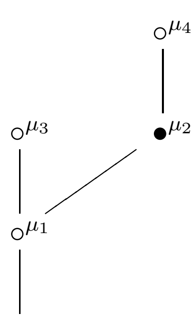
and $C(T) =$



as the result of the cut



Not an admissible cut is:



Remark 2.2.14. While admissible cuts split trees into branches and rooted trunk, leaf-restriction onto subsets I_1, I_2, where the set of leaves is a disjoint union of I_1 and

I_2, leads to a different type of splitting. This splitting is best adapted to trees which allow different labels only for leaves (and not for internal vertices), and is described as follows.

Definition 2.2.15. Let $M_0 = \{x_1, x_2, \ldots\}$, $M_1 = \emptyset$, $M_k = \{\bullet\}$(for $k \geq 2$). Let T be a planar admissibly labeled tree, especially T is reduced.

Given a split $\mathrm{Le}(T) = I_1 \uplus I_2$ of the set of leaves of T into two disjoint subsets I_1, I_2, we call the pair

$$\big(\mathrm{red}(T|I_1), \mathrm{red}(T|I_2)\big)$$

the induced leaf-split of T.

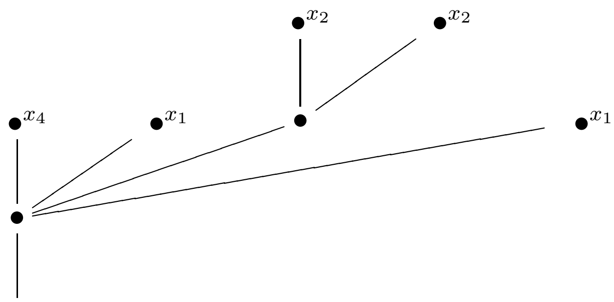
Given two planar admissibly labeled trees T^1, T^2 with n_1, n_2 leaves, and a planar tree T with $n_1 + n_2$ leaves, we say that T is a shuffle of T^1 and T^2, if

$$\mathrm{red}(T|I) = T^1, \mathrm{red}(T|I^c) = T^2$$

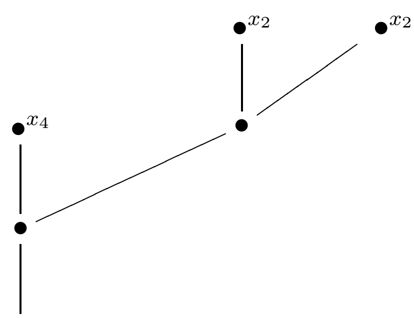
for some subset $I \subseteq \mathrm{Le}(T)$.

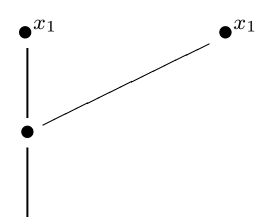
If $\mathrm{red}(T|I) = T^1, \mathrm{red}(T|I^c) = T^2$ (for some I) we call T^1 a complement of T^2 in T (and also T^2 a complement of T^1 in T).

Example 2.2.16. Consider the following planar tree T:

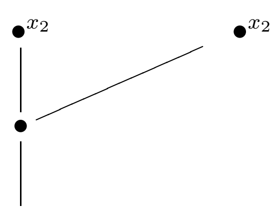


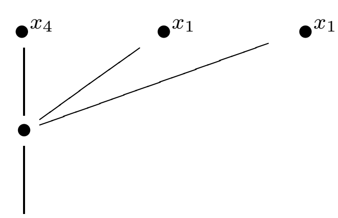
It is a shuffle of

$T^1 =$ 

and $T^2 =$ 

for the splitting given by the subsets $I = \{1, 3, 4\}$ and $I^c = \{2, 5\}$ of $\underline{5} = \mathrm{Le}(T)$.
The tree T is also a shuffle of the trees



and


2.3 Planar binary trees and Stasheff polytopes

We refine our notation for the various sets of trees, taking into account the numbers of leaves and internal vertices. Stasheff polytopes are helpful to link these sets.

We also have to introduce several operations for planar binary trees.

Definition 2.3.1. For $n, p \in \mathbb{N}$, let

$$\mathrm{PTree}_n := \{T \in \mathrm{PTree} : \#\mathrm{Ve}(T) = n\}$$

be the set of planar trees with n vertices, and let

$$\mathrm{PTree}^p := \{T \in \mathrm{PTree} : \#\mathrm{Le}(T) = p\}$$

be the set of planar trees with p leaves.

Furthermore, let

$$\mathrm{PTree}_n^p := \mathrm{PTree}_n \cap \mathrm{PTree}^p.$$

We make the analogous definitions for abstract trees, planar reduced trees, and the corresponding labeled sets.

Let

$$\mathrm{YTree}^p := \mathrm{YTree} \cap \mathrm{PTree}^p.$$

Remark 2.3.2. If we consider planar binary trees T and exclude the empty tree, the number p of leaves is one higher than the number of internal vertices. Thus $\#\mathrm{Ve}(T) = 2p - 1$.

Let $(M_k)_{k \in \mathbb{N}}$ be a collection of the form $M_0 = \{\circ\}$, $M_1 = \emptyset$, $M_2 \neq \emptyset$, $M_k = \emptyset$ (all $k \geq 3$).

Then we consider the labels from M_2 as labels of weight 1, and the labels from M_0 as labels of weight 0, and make the following definition.

Definition 2.3.3. Let $\mathrm{YTree}\{M_2\}$ be the set of non-empty admissibly labeled planar binary trees (i.e. the set of planar binary trees that are labeled at their internal vertices).

We define the degree of an element of $\mathrm{YTree}\{M_2\}$ to be the number of internal vertices, and set

$$\mathrm{YTree}^{(n)} := \mathrm{YTree}^{n+1}, \text{ all } n \geq 0,$$

and similarly

$$\mathrm{YTree}^{(n)}\{M_2\} := \mathrm{YTree}^{n+1}\{M_2\}, \text{ all } n \geq 0,$$

i.e. the image of $\mathrm{YTree}^{(n)}$ under deg is $\{n\}$.

We also define the degree of a planar binary forest to be the sum of the degrees of all its trees.

We identify $\mathrm{YTree}^{(n)}$ with the set $\mathrm{YTree}^{(n)}\{\bullet\}$, and we also use the notation

$$\mathrm{YTree}^{\infty} := \bigcup_{n \geq 0} \mathrm{YTree}^{(n)}.$$

Remark 2.3.4. The number $C_n = \#\mathrm{PRTree}^n$ of planar reduced trees with n leaves is called the n-th super-Catalan number.

The generating series for the super-Catalan numbers is $\frac{1}{4}(1 + x - \sqrt{1 - 6x + x^2})$ (cf. [Slo] A001003).

The first 10 super-Catalan numbers are

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049.$$

Definition 2.3.5. Given a planar binary tree T in YTree^∞ or in $\mathrm{YTree}^\infty\{M_2\}$, let $\alpha = \alpha(T)$ denote the first leaf of T (i.e. the leftmost leaf in a drawing which puts all leaves on one line).

Let $\omega = \omega(T)$ denote the last (i.e. rightmost) leaf of T.

For $1 \leq i < \#\mathrm{Le}(T)$, the i-th internal vertex of T is the highest internal vertex which belongs to both the paths from the i-th and the $(i+1)$-th leaf to the root.

Given a second planar binary tree S, we define

$$T \backslash S := T \circ_{\omega(T)} S.$$

Remark 2.3.6. The operation $\backslash$ is called under-operation and was introduced in [LR98]. Clearly $\backslash$ is associative and $S \backslash T \backslash Z$ is well-defined.

The analogous operation $\circ_\alpha$ given by $T \circ_\alpha S = T \circ_{\alpha(T)} S$ plays the role of an associative multiplication in a Hopf algebra defined by C. Brouder and A. Frabetti, see Section 3.6.

The opposite multiplication $\circ_\alpha^{op}$ is the over-operation $S/T := T \circ_{\alpha(T)} S$ of [LR98].

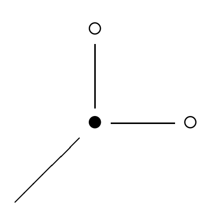
Using the mirror-operation $T \mapsto \bar{T}$, one can express S/T as $\overline{(\bar{T}) \backslash (\bar{S})}$.

The tree $|$ consisting of the root serves as a unit for all these operations, e.g. $S \backslash | = S = | \backslash S$.

Example 2.3.7. The (non-labeled) right comb of height n can be expressed as

$$\underbrace{Y \backslash Y \backslash Y \ldots \backslash Y}_{n}$$

if Y denotes the planar binary tree



Definition 2.3.8. Given $n \geq 1$ planar binary trees $T^1, \ldots, T^n$, and a sequence w of n labels from M_2, we define

$$\vee\!\!\nearrow_w(T^1 \ldots T^n)$$

to be the planar binary tree which can be obtained from the right comb C of height n as follows: We replace the first leaf $\alpha(C)$ by T^1, the second by T^2 and so on, leaving the $(n+1)$-th leaf (i.e. $\omega(C)$) unaltered. We just write $\vee\!\!\nearrow(T^1 \ldots T^n)$ if there is no choice of labels.

We denote the tree consisting of the root (which has got the standard label from $M_0 = \{\circ\}$) by $\overset{\circ}{|}$ or simply by $|$.

We define $\vee\!\!\nearrow(\emptyset) = |$.

Remark 2.3.9. The right comb of height n can then be written as

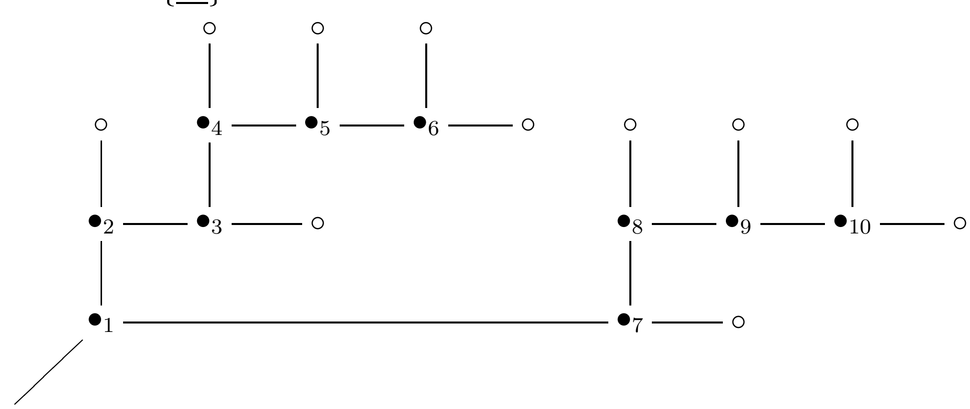
$$\vee\!\!\nearrow_w(n) := \vee\!\!\nearrow_w(\underbrace{| \ldots |}_{n}).$$

It is easy to see that the smallest set which contains $|$ and is closed under $\vee\!\!\nearrow_w$ operations contains all planar binary trees.

For every planar binary tree in $\mathrm{YTree}^{\infty}\{M_2\}$ this right comb presentation is unique. The left comb presentation is similarly defined. The right (or left) comb presentation induces a map φ_r (φ_l respectively) from planar binary trees to (not necessarily binary) planar forests such that

$$\varphi_r(|) = \emptyset, \quad \varphi_r\Big(\vee\!\!\nearrow_{w_1 \ldots w_n}(T^1 \ldots T^n)\Big) = \vee_{w_1}\big(\varphi_r(T^1)\big) \ldots \vee_{w_n}\big(\varphi_r(T^n)\big)$$

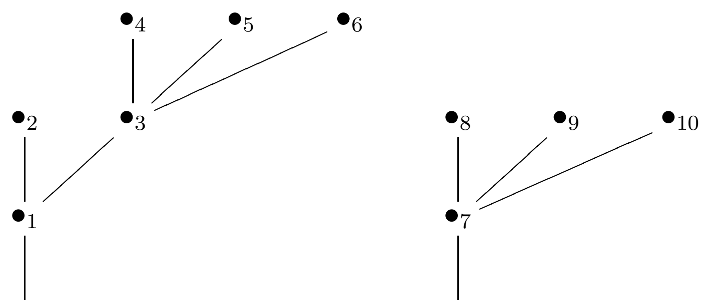
Example 2.3.10. The following admissibly labeled planar binary tree in $\mathrm{YTree}^{(10)}\{\underline{10}\}$



can be written in right comb presentation as

$$\vee\!\!\nearrow_{1,7}\Big(\vee\!\!\nearrow_{2,3}\big(|, \vee\!\!\nearrow_{4,5,6}(|||)\big)\ ,\ \vee\!\!\nearrow_{8,9,10}(|||)\Big)$$

Its image under φ_r is this forest:



Lemma 2.3.11. *Let the degree of a planar forest be given by the total number of vertices.*

Then the map φ_r (or φ_l) provides a bijection between the set $\mathrm{YTree}^{(n)}\{M_2\}$ *of degree n admissibly labeled planar binary trees and degree n planar forests labeled by the set M_2.*

The number of planar forests, labeled by the set M_2, of degree $n-1$, as well as the number of admissibly (M_2-)labeled planar binary trees of degree $n-1$ is given by

$$c_n \cdot (\#M_2)^{n-1}$$

where c_n is the n-th Catalan number.

The bijection between non-labeled planar trees with n vertices and planar binary trees with n leaves occurs as a special case.

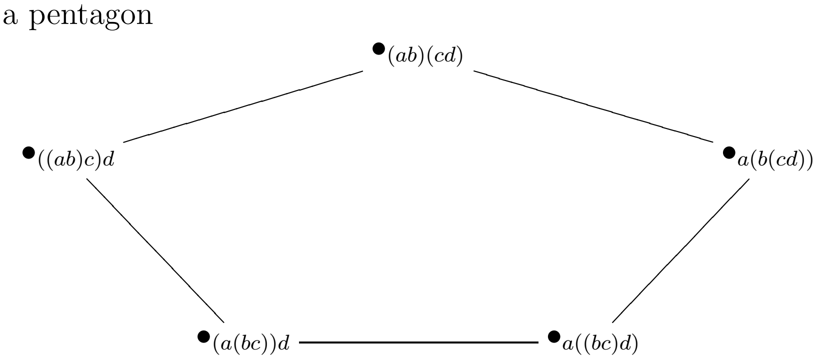
Proof. The Lemma is an easy consequence of our construction. $\square$

Remark 2.3.12. There is a convex polytope K_{n+1} of dimension $n-1$, $n \geq 1$, with one vertex for each planar binary tree with $n+1$ leaves, which is called the Stasheff polytope or associahedron in dimension $n-1$.

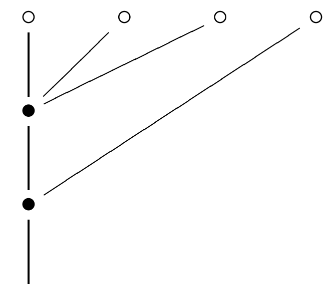
More exactly, K_{n+1} is a cell complex in dimension $n-1$ with the elements of $\mathrm{YTree}^{(n)} = \mathrm{PRTree}_{2n+1}^{n+1}$ as 0-cells.

The associahedra were created by J. Stasheff [Sta63] to study higher homotopies for associativity. If we consider the parenthesized strings (of 3 letters) given by the two planar binary trees with 3 leaves, we can get from one to the other by shifting a bracket, in other words (cf. [MSS], I.1), applying an associating homotopy $h(x, y, z)$ from $x(yz)$ to $(xy)z$.

In K_4 for example, the five planar binary trees with 4 leaves have to be arranged in a pentagon



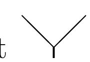
such that each side corresponds to an application of $h(x, y, z)$. These 5 sides can be labeled by the 5 reduced planar trees (with 4 leaves and 2 internal vertices) indicating the associating homotopy. For example, the edge between $(a(bc))d$ and $((ab)c)d$ corresponds to the tree



The $(n + 1)$-corolla represents the top dimensional cell of the polytope K_{n+1}.

By definition of the cell complex K_{n+1} the cells of dimension k are in bijection with the elements of $\mathrm{PRTree}^{n+1}_{2n-k+1}$, for $k = 0, \ldots, n - 1$.

Thus the super-Catalan numbers count all cells of the Stasheff polytope, and we can check that K_4, for example, has got 11 cells.

The polytope K_2 is a point which corresponds to the unique element  of $\mathrm{YTree}^{(1)} = \mathrm{PRTree}^2_3$, and K_3 is an interval.

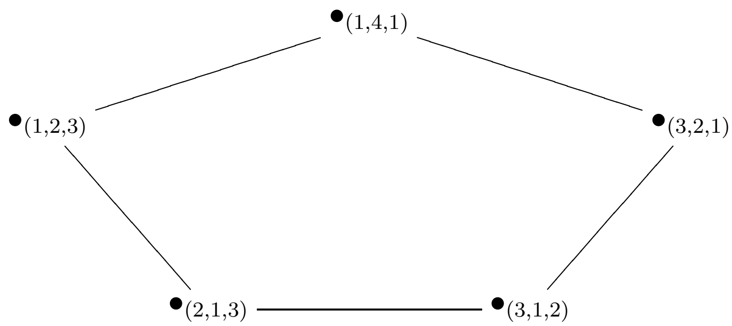
The facets (i.e. codimension one cells) of K_{n+1} are of the form $K_{r+1} \times K_{s+1}$, $r, s \geq 1$, $r + s = n$, with label obtained by grafting the s-corolla to the i-th leaf of the r-corolla, $1 \leq i \leq r$. One gets inclusion maps $\circ_i : K_{r+1} \times K_{s+1} \to K_{r+s+1}$ (cf. [MSS], I.1.6).

Pentagons and squares are the facets of K_5.

In fact, the realization of K_{n+1} as a convex polytope was an open problem at first. Several solutions were given (cf. [Sta97]). A simple realization, given in [Lod03a], associates to each planar binary tree $T \in \mathrm{YTree}^{(n)}$ a coordinate tuple $x(T)$ in (a hyperplane of) $\mathbb{R}^n$ as follows:

For $1 \leq i < n + 1 = \#\mathrm{Le}(T)$, consider the subtree with root given by the i-th internal vertex of T, and let a_i be the number of leaves on the left side, b_i the number of leaves on the right side (of the subtree's root). Then the i-th entry of $x(T)$ is given by $a_i b_i$.

For example, the coordinate tuples we get for K_4 are:



It is shown in [Lod03a], Theorem 1.1, that the convex hull of the points $x(T), T \in \mathrm{YTree}^{(n)}$, is a realization of the Stasheff polytope of dimension $n-1$.

It is possible to give an orientation to all the edges of the Stasheff polytope, see [LR02a]. When 0-cells are represented as parenthesized words, arrows are directed such that they correspond to shifting a bracket from left ((xx)x) to right (x(xx)).

The induced partial ordering on the set $\mathrm{YTree}^{(n)}$ (also called Tamari order) and the weak Bruhat order on the Coxeter group Σ_n are related by a projection $\Sigma_n \to \mathrm{YTree}^{(n)}$ given in [LR02a].

2.4 Free operads and quotients

The forgetful functor from the category of operads to the category Σ-Vect_K has a left adjoint, the free operad functor Γ (cf. [MSS], p.82, or [Fre03]).

Given a Σ-space A with basis $\alpha_1 \in A(n_1), \ldots, \alpha_l \in A(n_l)$ such that $n_i \geq 2$ for all i, we present an explicit construction of $\Gamma(A)$ in terms of reduced trees. We start with the case where we do not have to consider a Σ-action (i.e. the case of non-Σ operads or the case of regular operads).

It is possible to use abstract trees or planar trees for the description of free operads. Both approaches have advantages and disadvantages (cf. [MSS], p.83). A common way to give the definition of a concrete operad is to present it as a quotient of a free operad modulo an operad ideal.

Definition 2.4.1. Let $\underline{\mathcal{P}}$ (or $\mathcal{P}$, respectively) be a non-Σ operad (or an operad), and let $\mathcal{R}(n)$ be a subspace (invariant subspace) of $\mathcal{P}(n)$ for each n.

Then $\mathcal{R} = (\mathcal{R}(n))_{n\in\mathbb{N}}$ is called ideal of $\underline{\mathcal{P}}$ (or $\mathcal{P}$, respectively), if for $p \circ_i q$ defined in $\underline{\mathcal{P}}$ (or $\mathcal{P}$) it follows that $p \circ_i q \in \mathcal{R}$ whenever $p \in \mathcal{R}$ or $q \in \mathcal{R}$.

Remark 2.4.2. For every ideal $\mathcal{R}$ of an operad (or non-Σ operad) $\mathcal{P}$, the quotient $\mathcal{P}/\mathcal{R}$ given by $(\mathcal{P}/\mathcal{R})(n) = \mathcal{P}(n)/\mathcal{R}(n)$ is again a (non-Σ) operad.

Given a subset $\{r_i : i \in I\}$ of $\mathcal{P}$, the smallest ideal containing $\{r_i : i \in I\}$ is also called the operad ideal generated by the relations $r_i = 0$.

If we consider the free algebras over free operads $\mathcal{P}$, we can get the free algebras over a quotient operad $\mathcal{Q}$ as quotient algebras. In this way, the given operad ideals are determined by multilinear relations R in free algebras over free operads. One often

considers the class of all $\mathcal{Q}$-algebras as a subvariety determined by the identities R in the variety of all $\mathcal{P}$-algebras, cf. [Dre].

Definition 2.4.3. Let a collection $M = (M_k)_{k\geq 2}$, of sets be given, and set $M_0 := \{\circ\}, M_1 := \emptyset$.

Define for $n \geq 2$:

$$\Gamma(M)(n) = \bigoplus_{\text{planar trees } T \text{ with } n \text{ leaves}} K\ T\{(M_k)_{k\in\mathbb{N}}\}$$

and let $\Gamma(M)(0) = 0$, $\Gamma(M)(1) = K \cdot |$, where $|$ is the tree consisting of the root.

The elements of $\Gamma(M)(n)$ are linear combinations of admissibly labeled (necessarily reduced) planar trees with n leaves.

Let T^1 be such a tree with n leaves, and let T^2 be a further tree having m leaves. Then, for $i = 1, \ldots, n$,

$$T^1 \circ_i T^2$$

is given by the substitution of T^2 in T^1 at the i-th leaf, see Definition (2.1.10).

By K-linear extension, we define $\circ_i$-operations

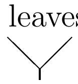
$$\circ_i : \Gamma(M)(n) \otimes \Gamma(M)(m) \to \Gamma(M)(m+n-1), \text{ all } n, m \geq 1,\ 1 \leq i \leq n.$$

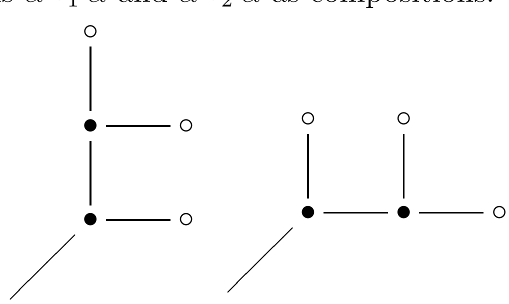
Lemma 2.4.4. *The sequence $\Gamma(M)$ of vector spaces, together with the unit $|$ and composition maps determined by the given $\circ_i$-operations, is the free non-Σ operad generated by the collection $M = (M_k)_{k\geq 2}$.*

Proof. This is a special case of the free operad construction (given for example in [MSS]); we give only a sketch of the proof. Free composition products of elements $\alpha = \alpha(x_1.x_2.\ldots.x_k) \in M_k$ $(k \geq 2)$ can be written as admissibly labeled leveled trees. We get vector spaces $\overline{\Gamma(M)}(n)$ generated by leveled trees with n leaves. The sequence $\Gamma(M)(n)$ is obtained by forgetting all levels, this corresponds exactly to the relations imposed by associativity of operad-composition. $\square$

Example 2.4.5. Let M_2 consist of one generator α, and let $M_k = \emptyset$ $(k \geq 3)$.

Then all elements of $\Gamma(M)$ are linear combinations of planar binary trees. For $n \geq 1$, we can identify a basis of $\Gamma(M)(n)$ with the set of (non-labeled) planar binary trees with n leaves. Especially, $\dim \Gamma(M)(n) = c_n$.

The tree  corresponds to the binary operation α, and we get ternary operations $\alpha \circ_1 \alpha$ and $\alpha \circ_2 \alpha$ as compositions:



This is the non-Σ operad $\underline{\mathcal{M}ag}$ of (non-unitary) magma algebras.

Example 2.4.6. Let each M_k, $k \geq 2$, consist of one generator $\vee^k$.

Then $\Gamma(M)(n)$ can be identified with the set of linear combinations of (non-labeled) planar trees with n leaves. The $\circ_i$-operations are exactly the $\circ_i$-operations defined for the the associahedron, see (2.3.12), given by grafting of trees.

Especially: the generator $\vee^k$ corresponds to the k-corolla, which can be considered as the grafting operation $\vee$ restricted to planar forests consisting of k trees (and their K-linear combinations).

This free non-Σ operad $\underline{\mathcal{K}}$, with $\underline{\mathcal{K}}^n = \Gamma(M)(n)$ (all n) is called the non-Σ operad of Stasheff polytopes. The cells of the associahedron in dimension $n-1$ form a basis of $\underline{\mathcal{K}}^{n+1}$. There is an obvious inclusion $\underline{\mathcal{M}ag} \to \underline{\mathcal{K}}$ defined, and we will denote $\underline{\mathcal{K}}$ also by $\underline{\mathcal{M}ag_\omega}$.

Remark 2.4.7. We sketch the construction of the free operad $\Gamma(A)$ for a given Σ-space A with basis $\alpha_1 \in A(n_1)$, …, $\alpha_l \in A(n_l)$ ($n_i \geq 2$ for all i). It is similar to the construction of the free non-Σ operad.

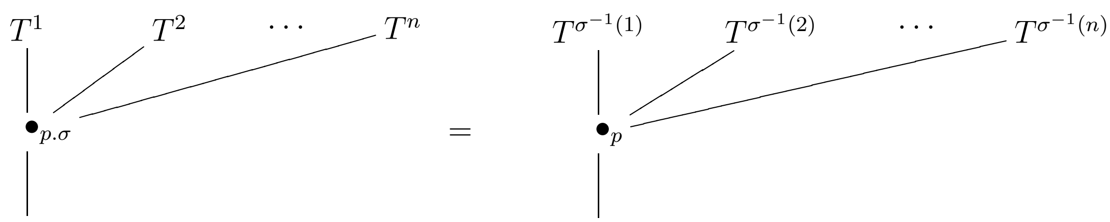
The sum

$$\bigoplus_{\text{planar trees } T \text{ with } n \text{ leaves}} K\ T\{(M_k)_{k\in\mathbb{N}}\}$$

has to be replaced by a sum over trees T with n leaves, labeled as follows:

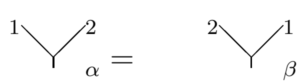
The labeling map sends the set of leaves bijectively to the set $\underline{n}$, and internal vertices of arity k are labeled by basis elements of $A(k)$.

The symmetric group Σ_n acts by permuting the labels of the leaves. For each internal vertex, we have to quotient out relations



In fact, this is a colimit-construction. (One can also obtain a quotient-free description using fixed representatives of abstract trees, cf. [Fre03], 3.6.1).

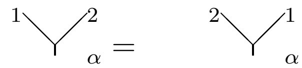
Example 2.4.8. Let $A = A(2)$ be the Σ_2-module given by two generators α, β with $\alpha.\tau = \beta$, where $\tau = (1,2)$ is the transposition. Then



and the free operad generated by A is the symmetrization $\mathcal{M}ag$ of $\underline{\mathcal{M}ag}$, with $\mathcal{M}ag(n) = \underline{\mathcal{M}ag}(n) \otimes_K K\Sigma_n$ (all n).

The symmetrization $\mathcal{M}ag_\omega$ of $\underline{\mathcal{M}ag_\omega}$ from (2.4.6) will be called the Stasheff operad, for short.

If $A = A(2)$ is the Σ_2-module given by one generator α with $\alpha.\tau = \alpha$, then



and

$$\Gamma(M)(n) = \bigoplus_{\text{abstract binary trees } T \text{ with } n \text{ leaves}} K \cdot T$$

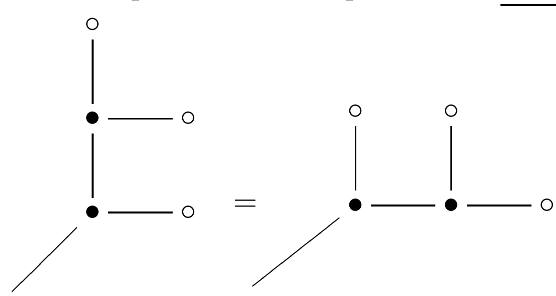
This operad is called the operad $\mathcal{C}mg$ of commutative magma algebras. It is not a regular operad.

Similarly, for the free operad Γ generated by $A(n) = K\{\alpha_n\}$ $(n \geq 2)$ with trivial action of Σ_n, we have

$$\Gamma(M)(n) = \bigoplus_{\text{abstract reduced trees } T \text{ with } n \text{ leaves}} K \cdot T$$

The operad $\mathcal{C}mg_\omega := \Gamma$ might be called the operad of commutative tree algebras.

Example 2.4.9. The non-Σ operad $\underline{\mathcal{A}s}$ is a quotient of $\underline{\mathcal{M}ag}$ with respect to the associativity relation



given by $\alpha \circ_1 \alpha = \alpha \circ_2 \alpha$.

For each n we get an identification of all operations with n arguments in the quotient.

Consequently, the operad $\mathcal{A}s$ (with $n!$ operations of n arguments) is the analogous quotient of $\mathcal{M}ag$.

Remark 2.4.10. Whenever there is a presentation by binary generators which are subject to quadratic relations, one speaks of a binary quadratic operad. More generally a quadratic operad is generated by (not necessarily binary) generators subject to relations that can be written as linear combinations of admissibly labeled trees with two internal vertices.

Then (using orthogonal relations) there is a quadratic dual operad $\mathcal{P}^!$, and there is a concept of Koszul duality, see [GK94].

We are not going to use the notion of Koszul operads, but it is an immediate observation that free operads are also Koszul operads.

Definition 2.4.11. (see [Lod01])

A dendriform algebra over K is a K-vector space A together with two binary operations $\prec, \succ \colon A \otimes A \to A$ called left and right, such that, if $x * y := x \succ y + x \prec y$ (for $x, y \in A$):

$$(x \prec y) \prec z = x \prec (y * z)$$
$$(x \succ y) \prec z = x \succ (y \prec z)$$
$$(x * y) \succ z = x \succ (y \succ z)$$

The non-Σ operad $\underline{\mathcal{D}end}$ of dendriform algebras is defined as the quotient of the free operad on two generators $\prec, \succ$ with respect to the quadratic relations induced by the three equations above.

The (regular) operad $\mathcal{D}end$ is the symmetrization of $\underline{\mathcal{D}end}$.

Remark 2.4.12. It is easy to check that the sum $*$ of the operations $\prec$ and $\succ$ is an associative operation (one sums up the lefthandsides and righthandsides of the three equations and gets $(x * y) * z = x * (y * z)$).

Free dendriform algebras will be described in Section 3.4.

A first example of dendriform algebras can be obtained as follows:

We equip the standard tensor coalgebra with the shuffle multiplication $\sqcup\!\sqcup$.

Then, for every pair of words $u = w_1.w_2 \ldots w_p$, $v = w_{p+1} \ldots w_r$, their product $u \sqcup\!\sqcup v = w_1.w_2 \ldots w_p \sqcup\!\sqcup w_{p+1} \ldots w_r$ is a finite sum, which can be 'splitted' in two parts: One defines $\prec$ by all shuffles with first letter w_1, and $\succ$ by all shuffles with first letter w_{p+1}, see [Lod01].

Remark 2.4.13. While operads describe collections of operations with many inputs but only one output, the concept of PROPs, cf. [Mac65], handles structures with many inputs and many outputs.

Since we are going to mention them sometimes in short remarks, we sketch how PROPs arise as generalizations of operads.

A PROP (compare [MV03]) is a sequence $\mathcal{P}$ of (Σ_m, Σ_n) bimodules together with two types of compositions, horizontal

$$\mathcal{P}(m_1, n_1) \otimes \ldots \otimes \mathcal{P}(m_s, n_s) \to \mathcal{P}(m_1 + \ldots + m_s, n_1 + \ldots + n_s),$$

for all $m_1, \ldots, m_s, n_1, \ldots, n_s \in \mathbb{N}^*$, and vertical

$$\mathcal{P}(m, n) \otimes \mathcal{P}(n, k) \to \mathcal{P}(m, k),$$

for all $m, n, k \in \mathbb{N}^*$.

A PROP is supposed to satisfy axioms which could be read off from the example of the endomorphism PROP given by:

$$\mathcal{E}nd_V(m, n) = \mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$$

with horizontal composition given by the tensor product of linear maps, and vertical composition by the ordinary composition of linear maps.

Similarly to algebras over operads, there are defined (al)gebras over PROPs. Many techniques for operads also hold for PROPs, e.g. Vallette (see [Val]) makes

use of Koszul duality for PROPs. In general, there is no free (al)gebra over a PROP defined, though.

We already noted in Remark (1.2.10) that operads are also defined for monoidal categories different from $(\mathrm{Vect}_K, \otimes)$ (cf. [MSS], II.1). This also holds for the concepts of PROPs.

Chapter 3

$\mathcal{P}$-Hopf algebras

3.1 Hopf algebra theory

Given two categories $\mathbf{C}, \mathbf{D}$, the study of representable functors from $\mathbf{C}$ to $\mathbf{D}$ leads to the study of co-$\mathbf{D}$ objects in $\mathbf{C}$, see [BH], [Abe]§A5, as their representing objects. Here we assume that the category $\mathbf{C}$ has finite categorical coproducts $A \sqcup_{\mathbf{C}} \ldots \sqcup_{\mathbf{C}} A$.

When $\mathbf{D}$ is the category of (semi-)groups, one speaks of co-group objects (or co-semigroup objects) A in $\mathbf{C}$. Being the representing object, A is provided with a morphism $\Delta : A \to A \sqcup_{\mathbf{C}} A$. The associativity of (semi-)group multiplication translates to coassociativity of Δ, i.e. the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \sqcup_{\mathbf{C}} A \arrow[d, "\Delta \sqcup_{\mathbf{C}} \mathrm{id}"] \\ A \otimes A \arrow[r, "\mathrm{id} \sqcup_{\mathbf{C}} \Delta"'] & A \sqcup_{\mathbf{C}} A \sqcup_{\mathbf{C}} A \end{tikzcd}$$

If $\mathbf{C}$ is the category of $\mathcal{C}om$-algebras, a representable group-valued functor is known as an affine group scheme, and a co-group (co-semigroup) object is known as a commutative Hopf algebra (or bialgebra).

In the category of $\mathcal{A}s$-algebras, the categorical coproduct is given by the free product of $\mathcal{A}s$-algebras. In Hopf algebra theory, we consider the category of $\mathcal{A}s$-algebras together with the tensor product $\otimes$ replacing the categorical coproduct.

Consequently, coassociativity of a K-linear map $\Delta : V \to V \otimes V$, for any vector space V, means that the diagram

$$\begin{tikzcd} V \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & V \otimes V \arrow[d, "\Delta \otimes \mathrm{id}"] \\ V \otimes V \arrow[r, "\mathrm{id} \otimes \Delta"'] & V \otimes V \otimes V \end{tikzcd}$$

commutes.

A coalgebra (C, Δ) in Hopf algebra theory just means a vector space C together with a coassociative K-linear map $\Delta : C \to C \otimes C$.

This agrees with the definition of an $\mathcal{A}s$-coalgebra in Definition (1.3.9). For each $n \geq 0$ there is (up to a constant factor) only one n-ary co-operation, called Δ^n. The map $\Delta^n : C \to C^{\otimes n}$ is given by

$$\Delta^0 = \mathrm{id}, \Delta^1 = \Delta$$
$$\Delta^n = (\Delta \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}) \circ \Delta^{n-1}$$

Usually one looks at counitary coalgebras, dually defined to unitary algebras:

The counit is a K-linear map $\varepsilon : C \to K$ with commutative diagram

$$\begin{tikzcd}
K \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \arrow[r, "\mathrm{id} \otimes \varepsilon"] & C \otimes K \arrow[dl, "="] \\
& C \arrow[u, "\Delta"] &
\arrow[from=1-1, to=2-2, "="']
\end{tikzcd}$$

If $g \in C$ is an element such that $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, then g is said to be group-like.

For g, h group-like elements, one defines (cf. [Mon], p.4)

$$\mathrm{Prim}_{g,h}(C) := \{c \in C : \Delta(c) = c \otimes g + h \otimes c\}$$

and calls its elements (g, h)-primitives.

Let $C = \cup_{i \in \mathbb{N}} C_i$ with finite-dimensional $C_i \subseteq C_{i+1}$ such that

$$\Delta(C_n) \subseteq \sum_{i=0}^{n} C_i \otimes C_{n-i} \text{ (all } n).$$

Then C is called a filtered coalgebra, cf.[Abe], p.91.

Setting $C^{(i)} = C_i / C_{i-1}$, one obtains a coalgebra $\oplus_{i \in \mathbb{N}} C^{(i)}$, which is called the graded coalgebra associated to $C = \cup_{i \in \mathbb{N}} C_i$.

A coalgebra is called graded, if $C = \oplus_{i \in \mathbb{N}} C^{(i)}$ and

$$\Delta(C^{(n)}) \subseteq \sum_{i=0}^{n} C^{(i)} \otimes C^{(n-i)} \text{ (all } n).$$

If furthermore there is only one group-like element g in C, $C^{(0)} = Kg \cong K$, and $C^{(1)} = \mathrm{Prim}_{g,g}(C)$ then C is said to be strictly graded.

A unitary associative algebra A together with the structure of a (counitary) coalgebra on A is called a bialgebra, if Δ and ε are algebra homomorphisms. The unit $1 = 1_A$ is group-like, and the elements of $\mathrm{Prim}_{1,1}(A)$ are called the primitive elements of the bialgebra A.

Considering bialgebras as co-semigroup objects with categorical coproduct replaced by $\otimes$, Hopf algebras $A = (A, \mu, \Delta, \eta, \varepsilon, \sigma)$ are $\otimes$-cogroup objects (in the category of unitary associative algebras). The $\otimes$-coinverse map, called antipode, is a K-linear map $\sigma : A \to A$ such that

$$\mu \circ (\sigma \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\mathrm{id} \otimes \sigma) \circ \Delta$$

(where $\eta : K \to A, 1 \mapsto 1_A$, and $\mu : A \otimes A \to A$ describe the unitary associative algebra structure).

Necessarily, $\sigma(1_A) = 1_A$, and $\sigma(a) = -a$ for every primitive element $a \in A$.

If $A_0 = A^{(0)} \cong K$ in the filtered case, with $\varepsilon : A \to K$ the augmentation map, an antipode can always be constructed recursively by

$$\sigma(a) = -a - \sum{}' \sigma\big(a_{(1)}\big) a_{(2)}$$

if $\sum' a_{(1)} \otimes a_{(2)}$ denotes $\Delta'(a) := \Delta(a) - a \otimes 1 - 1 \otimes a$.

Hence in the definition of filtered Hopf algebras it is not necessary to require an antipode.

In general, given a bialgebra A one can construct its Hopf envelope (see [Man], cf. also [GH97]). Or one can consider the completion $\hat{A}$ with respect to the topology induced by the augmentation ideal $I = \ker \varepsilon$, i.e. $\widehat{A} = \varprojlim(A/I^n)$, which is a complete Hopf algebra (cf. [Hol]).

For example, let $A = K\langle X_1, X_2, X_3\rangle$ be the free unitary $\mathcal{A}s$-algebra generated by variables X_1, X_2, X_3. Together with the algebra homomorphism Δ given by

$$\begin{aligned} &\Delta(X_1) = X_1 \otimes X_1, \quad \Delta(X_2) = X_2 \otimes X_2, \\ &\Delta(X_3) = X_1 \otimes X_3 + X_3 \otimes X_2, \end{aligned}$$

and with the counit defined by $\varepsilon(X_1) = 1 = \varepsilon(X_2)$, $\varepsilon(X_3) = 0$, A is a bialgebra.

Then the Hopf envelope is the algebra $K\langle X_1^{(i)}, X_2^{(i)}, X_3^{(i)} : i \geq 1\rangle/$ relations which assure that the anti-algebra homomorphism given by $\sigma(X_j^{(i)}) = X_j^{(i+1)}$ is an antipode.

The completion $\widehat{A}$ of A is the power series algebra $\widehat{F}_{\mathcal{A}s}(V_{\{x_1,x_2,x_3\}})$, where x_3 corresponds to X_3, but x_1 and x_2 correspond to $X_1 - 1$ and $X_2 - 1$. The counit sends $x_i \mapsto 0$. The comultiplication $\Delta : \widehat{A} \to \widehat{A} \hat{\otimes} \widehat{A}$ is given by

$$\begin{aligned} \Delta(x_i) &= x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i, \text{ for } i = 1, 2 \\ \Delta(x_3) &= x_3 \otimes 1 + 1 \otimes x_3 + x_1 \otimes x_3 + x_3 \otimes x_2. \end{aligned}$$

One easily constructs the antipode. It is the anti-algebra homomorphism given by

$$\begin{aligned} x_i &\mapsto \sum_{k=0}^{\infty} (-1)^k x_i^k \text{ (for } i = 1, 2) \\ x_3 &\mapsto \sum_{k=0}^{\infty} (-1)^{k+1} \Big(\sum_{j=0}^{k} x_1^j x_3 x_2^{k-j} \Big). \end{aligned}$$

3.2 Operads equipped with unit actions

Before we can generalize the notion of "usual" bialgebra and Hopf algebra structures on $\mathcal{C}om$-algebras or $\mathcal{A}s$-algebras to the setting of $\mathcal{P}$-algebras we need to require that the tensor product of $\mathcal{P}$-algebras is provided with the structure of a $\mathcal{P}$-algebra.

There are several approaches possible (cf.[Moe01], [Lod03b], [Hol01]). Here we follow [Lod03b].

Definition 3.2.1. Let $\mathcal{P}$ be an operad, $\mathcal{P}(0) = 0, \mathcal{P}(1) = K = K\,\mathrm{id}$.

Let a 0-ary element η be adjoined to the Σ-space $\mathcal{P}$ by $\mathcal{P}'(i) := \begin{cases} \mathcal{P}(i) & : i \geq 1 \\ K\eta & : i = 0. \end{cases}$

A unit action on $\mathcal{P}$ is a partial extension of the operad composition onto $\mathcal{P}'$ in the following sense:

We ask that composition maps $\mu_{n;m_1,\dots,m_n}$ (fulfilling the associativity, unitary, and invariance conditions) are defined on

$$\mathcal{P}'(n) \otimes \mathcal{P}'(m_1) \otimes \ldots \otimes \mathcal{P}'(m_n) \to \mathcal{P}'(m),\ m := m_1 + \ldots + m_n,$$

for all $m_j \geq 0$ $(j = 1, \dots, n)$, for $n \geq 2,\ m > 0$ (or $n \leq 1, m \geq 0$).

Especially defined are K-linear maps

$$\_ \circ_{n,i} (\eta) : \mathcal{P}'(n) \to \mathcal{P}'(n-1),\ n \geq 2,\ 1 \leq i \leq n.$$

Remark 3.2.2. Let a $\mathcal{P}$-algebra $\overline{A}$, with structure maps $\gamma(n)$, be given. Let $p \in \mathcal{P}(n)$. We denote $\gamma(0)(\eta) \in A := K \oplus \overline{A}$ by 1 (compare Definition (1.3.1)).

Thus the unit action gives sense to expressions

$$p(a_1, \dots, a_n) = \gamma(n)(p)(a_1 \otimes \ldots \otimes a_n)$$

in the case where some of the a_i are 1.

For $p \in \mathcal{P}(n), n \geq 2$,

$$p(1, \dots, 1)$$

is not defined. In Definition (3.2.1), the compositions of $p \otimes q_1 \otimes \ldots \otimes q_n$ might more generally be only required to exist if p is in a set M of fixed generators of $\mathcal{P}$.

In Section (1.3) we considered operads and operad algebras in the non-unitary case.

For example, the operad $\mathcal{C}om$ leads to commutative associative algebras without unit. Since there exists (up to a scalar factor) only one operation $\cdot_n$ in $\mathcal{P}(n)$ for each n, the canonical unit action is induced by $\cdot_n \circ_{n,i} (\eta) = \cdot_{n-1}$, $n \geq 2,\ 1 \leq i \leq n$. This just means that in any $\mathcal{C}om$-algebra A, occurrences of 1_A in products can be ignored.

Similarly, for regular operads that are generated by (single) binary operations, like $\mathcal{A}s$ and $\mathcal{M}ag$, there is a canonical unit action induced by $1 \cdot a = a \cdot 1 = a$.

In these cases, the notion of a unitary $\mathcal{P}$-algebra (defined as in definition (1.3.1), with $\mathcal{P}'$ in place of $\mathcal{P}$) makes sense.

Definition 3.2.3. If there are operations $\star_n \in \mathcal{P}(n)$, all $n \geq 2$, fulfilling

$$\star_n \circ_{n,i} (\eta) = \star_{n-1}, \text{ all } i$$
$$\star_2 \circ_{2,1} (\eta) = \star_2 \circ_{2,2} (\eta) = \mathrm{id}\,.$$

then we say that the unit action respects the operations $\star_n$, and we define

$$\mu_{n;0,\ldots,0}(\star_n \otimes \eta \otimes \ldots \otimes \eta) = \eta.$$

Definition 3.2.4. The unit action of (3.2.3) is said to be compatible with the relations of $\mathcal{P}$, if the relations still hold on $A = K\ 1 \oplus \overline{A}$, for every (non-unitary) $\mathcal{P}$-algebra $\overline{A}$, whenever the expressions are defined. We call $A = K\ 1 \oplus \overline{A}$ the unitary $\mathcal{P}$-algebra associated to $\overline{A}$, and we call the projection $K\ 1 \oplus \overline{A} \to K$ with kernel $\overline{A}$ the augmentation of A.

Proposition 3.2.5. *Let $\mathcal{R}$ be an operad ideal of $\mathcal{P} = \mathcal{A}s$ or $\mathcal{P} = \mathcal{M}ag$, given by relations $r_i, i \in I$. If the corresponding quotient operad $\mathcal{P}/(r_i : i \in I)$ is equipped with a compatible unit action, then also $\mathcal{P}/(r_i : i \in I)^n$ is equipped with a compatible unit action, for all $n \geq 1$.*

Here, $(r_i : i \in I)^n$ is the n-power of the ideal $(r_i : i \in I)$, generated by products of r_i (n factors).

Proof. We consider free unitary $\mathcal{P}$-algebras $F^1V = K1 \oplus F(V)$, and we factor out the relations induced by $(r_i : i \in I)^n$. Since the unit action for $\mathcal{P}$ is compatible with $(r_i : i \in I) \supseteq (r_i : i \in I)^n$, it is also compatible with $(r_i : i \in I)^n$. □

Definition 3.2.6. The unit action of (3.2.3) is called coherent, if for all $\mathcal{P}$-algebras $\overline{A}, \overline{B}$ it holds that $(\overline{A} \otimes K1) \oplus (K1 \otimes \overline{B}) \oplus (\overline{A} \otimes \overline{B})$ is again a $\mathcal{P}$-algebra with, for all p in some set of generators M of $\mathcal{P}$:

$$\begin{aligned}
&p(a_1 \otimes b_1, a_2 \otimes b_2, \ldots, a_n \otimes b_n) := \star_n(a_1, a_2, \ldots, a_n) \otimes p(b_1, b_2, \ldots, b_n)\\
&\quad \text{for } a_i \in A, b_i \in B, \text{ in case that at least one } b_j \in \overline{B},\\
&\text{and } p(a_1 \otimes 1, a_2 \otimes 1, \ldots, a_n \otimes 1) := p(a_1, a_2, \ldots, a_n) \otimes 1\\
&\quad \text{(if the righthandside is defined).}
\end{aligned}$$

Remark 3.2.7. The definition given here generalizes the definition in [Lod03b], §3.2., where $\mathcal{P}$ is assumed to be a binary quadratic operad (and where an associative operation $*$ plays the role of the operations $\star_n$).

The definition (3.2.6) implies that, given a coherent unit action, the tensor powers $A^{\otimes n}$ are defined as $\mathcal{P}$-algebras, because

$$\begin{aligned}
&p(a_1 \otimes (b_1 \otimes c_1), a_2 \otimes (b_2 \otimes c_2), \ldots, a_n \otimes (b_n \otimes c_n))\\
&= \star_n(a_1, a_2, \ldots, a_n) \otimes \star_n(b_1, b_2, \ldots, b_n) \otimes p(c_1, c_2, \ldots, c_n)\\
&= p((a_1 \otimes b_1) \otimes c_1, (a_2 \otimes b_2) \otimes c_2, \ldots, (a_n \otimes b_n) \otimes c_n)
\end{aligned}$$

The unit is $1 \otimes 1 \otimes \ldots \otimes 1$.

If $A = K \oplus F_{\mathcal{P}}(V) = K \oplus \bigoplus_{n=1}^{\infty} \mathcal{P}(n) \otimes_{\Sigma_n} V_X^{\otimes n}$ is a free $\mathcal{P}$-algebra, then $A^{\otimes n}$ is as a $\mathcal{P}$-algebra (non-freely) generated by the elements

$$x_i^{\otimes n,p} := \underbrace{1 \otimes \ldots \otimes 1}_{p-1} \otimes x_i \otimes \underbrace{1 \otimes \ldots \otimes 1}_{n-p} \quad (\text{for } 1 \leq p \leq n, \ \text{all } x_i \in X).$$

Example 3.2.8. 1) The operads $\mathcal{C}om, \mathcal{A}s$ are equipped with canonical unit actions, which are clearly coherent.

2) In the same way, the operad $\mathcal{M}ag$ is equipped with a canonical unit action, and we are going to define a similar unit action on $\mathcal{M}ag_\omega$.

3) The unit actions on $\mathcal{M}ag$ and $\mathcal{M}ag_\omega$ induce unit actions on $\mathcal{C}mg$ and $\mathcal{C}mg_\omega$.

4) The operad $\mathcal{D}end$ is also equipped with a coherent unit action, see Section (3.4).

5) If $\mathcal{R}$ is an operad ideal of $\mathcal{P} = \mathcal{A}s$ or $\mathcal{P} = \mathcal{M}ag$, given by relations $r_i, i \in I$ as in Proposition (3.2.5), it is easy to see that $\mathcal{P}/[(r_i : i \in I), (r_i : i \in I)]$ is equipped with a coherent unit action, too. (The commutator ideal may also be replaced by the 'associator ideal' (R, R, R) or combinations of these ideals.)

Remark 3.2.9. The structure of a $\mathcal{P}$-algebra on $F_{\mathcal{P}}(V) \otimes F_{\mathcal{P}}(V)$ leads to an operad morphism $\mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$, which is coassociative. Vice versa such an operad morphism determines the $\mathcal{P}$-algebra structure on the tensor product (as noted by [Foi04], [vdL]). There is a corresponding notion of Hopf algebras over Hopf operads, cf. [Moe01] and [GJ94], which does not cover the case of $\mathcal{D}end$-algebras.

3.3 The definition of $\mathcal{P}$-Hopf algebras

Since Hopf algebras combine operations and cooperations, there is no operad whose algebras or coalgebras are Hopf algebras. To describe them, and also generalizations with not necessarily associative operations and not necessarily associative cooperations, one would use PROPs, see Remark (2.4.13).

Here we are only interested in the case, where the set of cooperations is generated by one coassociative cooperation. Thus we do not need the generality of PROPs and stay close to operad theory.

Definition 3.3.1. Let $\mathcal{P}$ be equipped with a coherent unit action and let $A = K\,1 \oplus \overline{A}$ be a unitary $\mathcal{P}$-algebra. Let $\Delta : A \to A \otimes A$ be a K-linear coassociative map (called comultiplication map), such that $\Delta(1) = 1 \otimes 1$ and $\Delta(a) - a \otimes 1 - 1 \otimes a \in \overline{A} \otimes \overline{A}$ for all $a \in \overline{A}$.

Then A together with Δ is called an (augmented) $\mathcal{P}$-bialgebra, if Δ is a morphism of unitary $\mathcal{P}$-algebras, i.e. if $\Delta \circ \mu_A = \mu_{A \otimes A} \circ (\Delta \otimes \ldots \otimes \Delta)$, for all $\mu \in \mathcal{P}$.

Remark 3.3.2. In the general case, the condition that Δ is a morphism of $\mathcal{P}$-algebras does not imply that the operations μ_A (for $\mu \in \mathcal{P}$) are morphisms of $\mathcal{P}$-coalgebras. In fact, we do not have provided $A \otimes A$ with the structure of a $\mathcal{P}$-coalgebra in general.

Definition 3.3.3. Elements a of an (augmented) $\mathcal{P}$-bialgebra A are called primitive, if

$$\Delta(a) - a \otimes 1 - 1 \otimes a = 0.$$

Definition 3.3.4. Let $A = \cup_{n \in \mathbb{N}} A_n$ be an (augmented) $\mathcal{P}$-bialgebra, with comultiplication Δ. We assume that each A_i is a finite dimensional vector space, and that $A_i \subseteq A_{i+1}$ (all i).

Then A is called a filtered $\mathcal{P}$-Hopf algebra if

$$\Delta(A_n) \subseteq \sum_{i=0}^{n} A_i \otimes A_{n-i} \text{ (all } n).$$

Let $\Delta'(a) := \Delta(a) - a \otimes 1 - 1 \otimes a$.

A filtered $\mathcal{P}$-Hopf algebra $A = \cup_{n \in \mathbb{N}} A_n$ is called connected, if $A_0 = K \cdot 1$, and (for $n \geq 1$)

$$A_n = \Big\{ a \in A : \Delta'(a) \in \sum_{i=1}^{n-1} A_i \otimes A_{n-i} \Big\}.$$

We call a filtered $\mathcal{P}$-Hopf algebra $A = \cup_{n \in \mathbb{N}} A_n$ a strictly graded $\mathcal{P}$-Hopf algebra, if

$$\Delta'(A^{(n)}) \subseteq \sum_{i=1}^{n-1} A^{(i)} \otimes A^{(n-i)} \text{ (all } n \geq 1)$$

for $A^{(i)} \subseteq A_i$ such that $A = \oplus_{n \in \mathbb{N}} A^{(n)}$, with $A^{(0)} = K$.

Homomorphisms of $\mathcal{P}$-bialgebras and $\mathcal{P}$-Hopf algebras $(A, \Delta_A), (B, \Delta_B)$ are unitary $\mathcal{P}$-algebra homomorphisms $\varphi : A \to B$ with $\varphi(\overline{A}) \subseteq \overline{B}$ and

$$\Delta_B \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_A.$$

Remark 3.3.5. In (3.3.4) we used the term filtered $\mathcal{P}$-Hopf algebra instead of filtered $\mathcal{P}$-bialgebra.

If μ is a binary operation of $\mathcal{P}$, A is a connected filtered $\mathcal{P}$-Hopf algebra, and $\mu\big(\cup_{k=0}^{n} A_k \otimes A_{n-k}\big) \subseteq A_n$, then one can construct recursively a K-linear map σ_l by

$$\sigma_l(a) = -a - \mu \circ \big(\sigma_l \otimes \mathrm{id}\big) \circ \Delta'(a)$$

and a K-linear map σ_r by

$$\sigma_r(a) = -a - \mu \circ \big(\mathrm{id} \otimes \sigma_r\big) \circ \Delta'(a).$$

The maps σ_l, σ_r might be called left and right antipodes. (They do not have to coincide, see Remark (4.3.4)).

Example 3.3.6. Let $\mathcal{P}$ be equipped with a coherent unit action and let A be the free unitary $\mathcal{P}$-algebra generated by a vector space V. Thus there is a $\mathcal{P}$-algebra homomorphism Δ_a defined by

$$\Delta_a(v) = v \otimes 1 + 1 \otimes v, \text{ all } v \in V$$

We call Δ_a the diagonal or co-addition.

Since $(\Delta_a \otimes \mathrm{id})\big(\Delta_a(v)\big) = v \otimes 1 \otimes 1 + 1 \otimes v \otimes 1 + 1 \otimes 1 \otimes v = (\mathrm{id} \otimes \Delta_a)\big(\Delta_a(v)\big)$ for all $v \in V$, the $\mathcal{P}$-algebra homomorphisms $(\Delta_a \otimes \mathrm{id}) \circ \Delta_a$ and $(\mathrm{id} \otimes \Delta_a) \circ \Delta_a$ coincide, and we get a strictly graded $\mathcal{P}$-Hopf algebra.

The name co-addition is chosen in analogy to the usage in [BH], where any abelian group valued representable functors leads to a co-addition (on the representing object).

Definition 3.3.7. Let $\mathcal{P}$ be equipped with a coherent unit action such that for every unitary $\mathcal{P}$-algebra A the K-linear map $\tau : A \otimes A \to A \otimes A, a_1 \otimes a_2 \mapsto a_2 \otimes a_1$ is a $\mathcal{P}$-algebra isomorphism.

Then, given a unitary $\mathcal{P}$-algebra A, we call a coassociative $\mathcal{P}$-algebra homomorphism $\Delta : A \to A \otimes A$ cocommutative, if $\tau \circ \Delta = \Delta$.

Remark 3.3.8. By definition, whenever $\mathcal{P}$ is equipped with a coherent unit action as in Definition (3.3.7), the co-addition Δ_a is cocommutative.

If $\mathcal{P}$ is $\mathcal{C}om$ or $\mathcal{A}s$, this is the classical definition. We are going to see that $\mathcal{D}end$ does not fulfill the condition.

The cocommutative Hopf algebra given by the free unitary $\mathcal{A}s$-algebra $K\langle X\rangle$ together with the diagonal Δ_a is well-known. Dually one can equip the standard tensor coalgebra (i.e. $K\langle X\rangle$ with its deconcatenation coalgebra-structure, see Example (1.3.11)) with the shuffle multiplication $\sqcup\!\sqcup = \Delta_a^*$, which is commutative (cf. [Reu], Chapter 1).

Definition 3.3.9. For any strictly graded $\mathcal{P}$-Hopf algebra $A = \bigoplus_{n\in\mathbb{N}} A^{(n)}$, we define the vector space

$$A^{*g} = \bigoplus_{n\in\mathbb{N}} (A^{*g})^{(n)} = \bigoplus_{n\in\mathbb{N}} (A^{(n)})^*$$

where $V^* = \mathrm{Hom}_K(V, K)$ for any vector space V.

We call A^{*g} the graded dual of A.

We denote by $\Delta^* : A^{*g} \otimes A^{*g} \to A^{*g}$ the K-linear map given by $\big(\Delta^*(f_1 \otimes f_2)\big)(a) = f_1(a_{(1)}) f_2(a_{(2)})$, where $f_1, f_2 \in A^{*g}$ and $a \in A$ with $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

The maps Δ^* and Δ are adjoint with respect to the canonical bilinear form $\langle , \rangle : A^{*g} \times A \to K$, i.e.

$$\langle \Delta^*(f_1 \otimes f_2), a\rangle = \langle f_1 \otimes f_2, \Delta(a)\rangle,$$

where $\langle f_1 \otimes f_2, g_1 \otimes g_2\rangle = \langle f_1, g_1\rangle \otimes \langle f_2, g_2\rangle$.

Lemma 3.3.10. *The space A^{*g} together with the operations μ^*, $\mu \in \mathcal{P}$, is a $\mathcal{P}$-coalgebra in the sense of Definition (1.3.9).*

The space A^{*g} together with $\Delta^* : A^{*g} \otimes A^{*g} \to A^{*g}$ is a graded $\mathcal{A}s$-algebra with unit $1 \in (A^{*g})^{(0)} = K$.

Together with its induced $\mathcal{P}$-Hopf algebra structure, $(A^{*g})^{*g}$ is isomorphic to A.

Proof. This is just the classical result, modified using the fact that a $\mathcal{P}$-algebra structure on A induces a $\mathcal{P}$-coalgebra structure on A^{*g}.

□

Remark 3.3.11. In the special case where $\mathcal{P} = \mathcal{A}s$, the graded dual (A^{*g}, Δ^*) of a strictly graded Hopf algebra (A, Δ) is again a strictly graded Hopf algebra. Furthermore, a cocommutative Δ leads to a $\mathcal{C}om$-Hopf algebra (A^{*g}, Δ^*); and a $\mathcal{C}om$-Hopf algebra (A, Δ) leads to a cocommutative Δ^*.

3.4 $\mathcal{D}end$-Hopf algebras of Loday and Ronco

We are going to recall the construction of the free $\mathcal{D}end$-algebra $(K\mathrm{YTree}^\infty, \prec, \succ)$ from [Lod01], and we state some of its properties.

The $\mathcal{D}end$-algebra $(K\mathrm{YTree}^\infty, \prec, \succ)$ can be provided with the structure of a $\mathcal{D}end$-Hopf algebra. Since the unit 1 plays the role of the planar binary tree which consists of the root, 1 is usually denoted as $|$.

Lemma 3.4.1. *(see [Lod03b])*

For any $\mathcal{D}end$-algebra $\overline{A}$, $x, y \in \overline{A}$, let

$$| \prec y := 0, \ x \prec | := x, \ x \succ | := 0, \ | \succ y := y$$
$$| \prec | \text{ and } | \succ | \text{ are not defined.}$$

The induced unit action for $\mathcal{D}end$ is coherent.

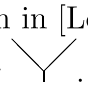
Example 3.4.2. (cf. [LR98], [Lod03b])

Consider the vector space $K\mathrm{YTree}^\infty$ together with operations $\prec$ and $\succ$, which are recursively defined as follows:

For T, Z planar binary trees $\neq |$, we set

$$T \prec Z = T^l \vee (T^r * Z), \ T \succ Z = (T * Z^l) \vee Z^r$$

where $|$ is neutral for the associative operation $*$ (the sum of $\prec$ and $\succ$).

It is shown in [Lod01] that $(K\mathrm{YTree}^\infty, \prec, \succ)$ is the free unitary $\mathcal{D}end$-algebra on one generator .

We consider $(K\mathrm{YTree}^\infty, *)$ as a graded associative algebra, graded with respect to the canonical degree function induced by

$$\deg T = n \text{ for } T \in \mathrm{YTree}^{(n)}.$$

The associative algebra $(K\mathrm{YTree}^\infty, *)$ is a free associative algebra with basis $\mathbb{V}(T), T \in \mathrm{YTree}^\infty$, cf. [LR98].

The free $\mathcal{D}end$-algebra generated by $\curlyvee$ can be provided with a $\mathcal{D}end$-Hopf algebra structure.

Lemma 3.4.3. *([Lod03b])*

The free $\mathcal{D}end$-algebra generated by $\curlyvee$ is a $\mathcal{D}end$-Hopf algebra with Δ determined by

$\Delta(\curlyvee) = \curlyvee \otimes | + | \otimes \curlyvee$.

For T, Z planar binary trees, it holds that

$$\Delta(T \vee Z) = T \vee Z \otimes | + \sum T_{(1)} * Z_{(1)} \otimes T_{(2)} \vee Z_{(2)}.$$

Proof. The first assertion is clear. For the second, we observe that

$$\begin{aligned}
\Delta(T \vee Z) &= \Delta(T \succ \curlyvee \prec Z) = \Delta(T) \succ (\curlyvee \otimes | + | \otimes \curlyvee) \prec \Delta(Z) \\
&= \sum (T_{(1)} \otimes T_{(2)}) \succ (\curlyvee \otimes |) \prec (Z_{(1)} \otimes Z_{(2)}) + \sum T_{(1)} * Z_{(1)} \otimes T_{(2)} \succ \curlyvee \prec Z_{(2)} \\
&= T \vee Z \otimes | + \sum T_{(1)} * Z_{(1)} \otimes T_{(2)} \vee Z_{(2)},
\end{aligned}$$

because the second tensor component of $(T_{(1)} \otimes T_{(2)}) \succ (\curlyvee \otimes |) \prec (Z_{(1)} \otimes Z_{(2)})$ is zero if not $T_{(2)}, Z_{(2)} \in K$, and $T_{(2)}, Z_{(2)} \in K$ (inductively) implies $T_{(1)} = T$, $Z_{(1)} = Z$,

$(T_{(1)} \otimes T_{(2)}) \succ (\curlyvee \otimes |) \prec (Z_{(1)} \otimes Z_{(2)}) = T \succ \curlyvee \prec Z \otimes |$.

□

Remark 3.4.4. In [Ron00a],[Ron02] Ronco has defined non-unital dendriform Hopf algebras as $\mathcal{D}end$-algebras A together with a K-linear coassociative map $\Delta' : A \otimes A \to A$, such that:

$$\begin{aligned}
\Delta'(x \succ y) &= x \otimes y + \sum x_{(1)} * y_{(1)} \otimes x_{(2)} \succ y_{(2)} + \sum x * y_{(1)} \otimes x_{(2)} \\
&\quad + \sum y_{(1)} \otimes x \succ y_{(2)} + \sum x_{(1)} \otimes x_{(2)} \succ y \\
\Delta'(x \prec y) &= y \otimes x + \sum x_{(1)} * y_{(1)} \otimes x_{(2)} \prec y_{(2)} + \sum x_{(1)} * y \otimes x_{(2)} \\
&\quad + \sum y_{(1)} \otimes x \prec y_{(2)} + \sum x_{(1)} \otimes x_{(2)} \prec y
\end{aligned}$$

After adjoining a unit $|$ to the non-unital dendriform Hopf algebra, the map Δ given by $\Delta(|) = | \otimes |$, $\Delta(T) = T \otimes | + | \otimes T + \Delta'(T)$, $T \neq |$, is easily seen to be a morphism of $\mathcal{D}end$-algebras (in the sense above). For example, to check the formula for $\Delta'(x \prec y)$ one can calculate $\Delta(x) \prec \Delta(y) - x \prec y \otimes | - | \otimes x \prec y$ in the unitary case as an expression consisting of seven terms: Two of these terms, namely the terms $(x \otimes |) \prec (| \otimes y)$ and $(x \otimes |) \prec \Delta'(y)$, are zero.

There is an obvious forgetful functor from $\mathcal{D}end$-Hopf algebras to $\mathcal{A}s$-Hopf algebras.

Definition 3.4.5. The $\mathcal{A}s$-Hopf algebra structure on $\big(K\mathrm{YTree}^\infty, *\big)$ given by the formula of (3.4.3), i.e. with $\Delta_{LR}(|) = | \otimes |$ and

$$\Delta_{LR}(T \vee Z) = T \vee Z \otimes | + \sum T_{(1)} * Z_{(1)} \otimes T_{(2)} \vee Z_{(2)}$$

for T, Z planar binary trees, is called the Hopf algebra of Loday and Ronco, see [LR98].

We write $\Delta_{LR}(T) = T \otimes 1 + 1 \otimes T + \Delta'_{LR}(T)$.

Lemma 3.4.6. *It holds that*

(i)

$$\Delta_{\mathrm{LR}}(\mathbb{V}T) = \mathbb{V}T \otimes | + \sum T_{(1)} \otimes \mathbb{V}(T_{(2)})$$

(ii)

$$\Delta_{\mathrm{LR}}(T \vee |) = T \vee | \otimes | + \sum T_{(1)} \otimes T_{(2)} \vee |$$

(iii)

$$\begin{aligned}&\Delta_{\mathrm{LR}}(\vee\!\!\nearrow(T^1.T^2 \dots T^n)) \\ &\qquad = \sum_{j=0}^{n} T^1_{(1)} * \dots * T^j_{(1)} * \vee\!\!\nearrow(T^{j+1} \dots T^n) \otimes \vee\!\!\nearrow(T^1_{(2)}.T^2_{(2)} \dots T^j_{(2)})\end{aligned}$$

Proof. Assertions (i) and (ii) follow directly from the formula of (3.4.3).

For (iii) we use induction, where the case $n = 1$ is assertion (ii).

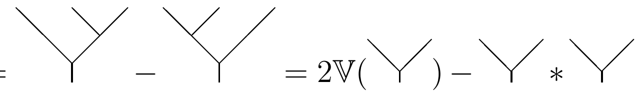
For $j = 0$,

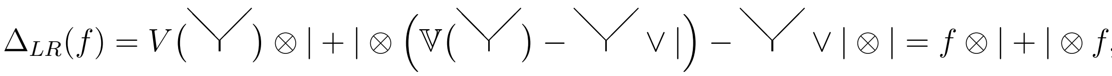
$T^1_{(1)} * \dots * T^j_{(1)} * \vee\!\!\nearrow(T^{j+1} \dots T^n) \otimes \vee\!\!\nearrow(T^1_{(2)} \dots T^j_{(2)}) = \vee\!\!\nearrow(T^1 \dots T^n) \otimes |$.

Since $\vee\!\!\nearrow(T^1 \dots T^n) = T^1 \vee \big(\vee\!\!\nearrow(T^2 \dots T^n)\big)$,

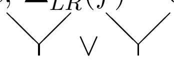
$$\begin{aligned}&\Delta_{\mathrm{LR}}(\vee\!\!\nearrow(T^1 \dots T^n)) - \vee\!\!\nearrow(T^1 \dots T^n) \otimes | = (*, \vee)\big(\Delta(T^1) \otimes \Delta(\vee\!\!\nearrow(T^2 \dots T^n))\big) \\ &= \sum_{j=1}^{n} T^1_{(1)} * \dots * T^j_{(1)} * \vee\!\!\nearrow(T^{j+1} \dots T^n) \otimes \vee\!\!\nearrow(T^1_{(2)} \dots T^j_{(2)}).\end{aligned}$$

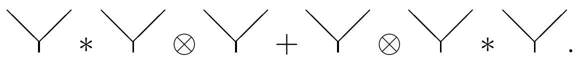
$\square$

Example 3.4.7. For  we compute

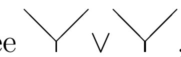


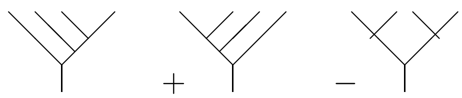
i.e. f is primitive, $\Delta'_{LR}(f) = 0$.

The image of  under Δ'_{LR} is



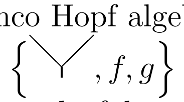
It follows from formula (iii) that $\Delta_{\mathrm{LR}}(\vee\!\!\nearrow\!(n)) = \sum_{j=0}^{n} \vee\!\!\nearrow\!(n-j) \otimes \vee\!\!\nearrow\!(j)$. An analogous formula holds for left combs.

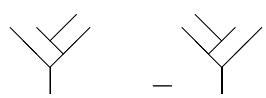
Combining the combs of height 3 and the tree , we get that the element g given by



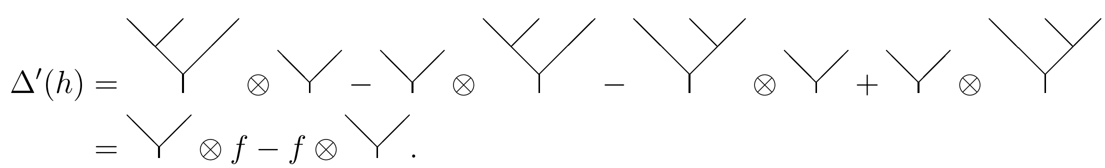
is primitive.

The Loday-Ronco Hopf algebra is not cocommutative.

We can extend  to an algebra basis of $(K\mathrm{YTree}^{\infty}, *)$, by adding one non-primitive element h of degree 3 and further elements of degree ≥ 4. For example, set

$h :=$ 

with



3.5 The Connes-Kreimer Hopf algebra of renormalization

Definition 3.5.1. Let $K\langle \mathrm{PTree} \rangle$ be graded with respect to the canonical degree function induced by

$$\deg T = n = \#\mathrm{Ve}(T), \text{ for } T \in \mathrm{PTree}_n.$$

The empty tree $\emptyset \in \mathrm{PTree}' = \mathrm{PTree} \cup \{\emptyset\}$ is identified with the unit of $K\langle\mathrm{PTree}\rangle$.

Proposition 3.5.2. *Let $\underline{\Delta} : K\langle\mathrm{PTree}\rangle \to K\langle\mathrm{PTree}\rangle \otimes K\langle\mathrm{PTree}\rangle$ be the graded unitary algebra homomorphism which maps any tree $T = \vee(T^1 \dots T^n) \in \mathrm{PTree}$ to*

$$\underline{\Delta}(T) = T \otimes \emptyset + \sum T^1_{(1)} \dots T^n_{(1)} \otimes \vee\big(T^1_{(2)} \dots T^n_{(2)}\big),$$

where we can assume that the images $\underline{\Delta}(T^k) = \sum T^k_{(1)} \otimes T^k_{(2)}$ of $T^k \in \mathrm{PTree}' = \mathrm{PTree} \cup \{\emptyset\}$, $k = 1, \dots, n$, are already defined.
Then $(K\langle\mathrm{PTree}\rangle, \underline{\Delta})$, together with the augmentation map as counit is a (graded) Hopf algebra.

Proof. (Cf. also [Foi].)
It is easy to check, that $\sum T^1_{(1)} \dots T^n_{(1)} \otimes \vee\big(T^1_{(2)} \dots T^n_{(2)}\big)$ is given by $\emptyset \otimes T$ plus terms $a_{(1)} \otimes a_{(2)}$ with both $a_{(i)}$ of order ≥ 1, thus the augmentation map is a counit. We have to show that $\underline{\Delta}$ is coassociative.
As $\underline{\Delta}(\emptyset) = \emptyset \otimes \emptyset$, $\big((\underline{\Delta} \otimes \mathrm{id})\underline{\Delta} - (\mathrm{id} \otimes \underline{\Delta})\underline{\Delta})\big)(\emptyset) = 0$.
Let T be a tree. We can assume that $\big((\underline{\Delta} \otimes \mathrm{id})\underline{\Delta} - (\mathrm{id} \otimes \underline{\Delta})\underline{\Delta})\big)(Z) = 0$ for all trees Z of a degree smaller than $\deg(T)$.
Now $\underline{\Delta} \otimes \mathrm{id}$ maps
$T \otimes \emptyset + (\cdot, \vee)\big(\underline{\Delta}(T^1) \otimes \dots \otimes \underline{\Delta}(T^n)\big) := T \otimes \emptyset + \sum T^1_{(1)} \dots T^n_{(1)} \otimes \vee\big(T^1_{(2)} \dots T^n_{(2)}\big)$ onto

$$\begin{aligned}&\Big(T \otimes \emptyset + (\cdot, \vee)\big(\underline{\Delta}(T^1) \otimes \dots \otimes \underline{\Delta}(T^n)\big)\Big) \otimes \emptyset \\ &+ (\cdot, \cdot, \vee)\big((\underline{\Delta} \otimes \mathrm{id})\underline{\Delta}(T^1), \dots, (\underline{\Delta} \otimes \mathrm{id})\underline{\Delta}(T^n)\big).\end{aligned}$$

On the other hand, $\mathrm{id} \otimes \underline{\Delta}$ maps the same element $T \otimes \emptyset + (\cdot, \vee)\big(\underline{\Delta}(T^1) \otimes \dots \otimes \underline{\Delta}(T^n)\big)$ onto

$$\begin{aligned}T \otimes \emptyset \otimes \emptyset &+ \sum T^1_{(1)} \dots T^n_{(1)} \otimes \underline{\Delta} \circ \vee\big(T^1_{(2)} \dots T^n_{(2)}\big) \\ &= \Big(T \otimes \emptyset + (\cdot, \vee)\big(\underline{\Delta}(T^1) \otimes \dots \otimes \underline{\Delta}(T^n)\big)\Big) \otimes \emptyset \\ &\qquad + (\cdot, \cdot, \vee)\big((\mathrm{id} \otimes \underline{\Delta})\underline{\Delta}(T^1), \dots, (\mathrm{id} \otimes \underline{\Delta})\underline{\Delta}(T^n)\big).\end{aligned}$$

By induction, it follows that $\underline{\Delta}$ is coassociative.

□

Remark 3.5.3. In [Kre98], Kreimer discovered a $\mathcal{C}om$-Hopf algebra for the use of renormalization of quantum field theories. It was further studied by Connes and Kreimer as a $\mathcal{C}om$-Hopf algebra structure on $K[\mathrm{ATree}]$, cf. [CK98], [CK00], and [Kre02].

The Connes-Kreimer Hopf algebra is obtained from $(K\langle\mathrm{PTree}\rangle, \underline{\Delta})$ by considering the quotient Hopf algebra with respect to the ideal generated by all commutators and 'forgetting' the planar structure of the trees. Its comultiplication Δ_{CK} is the unitary algebra homomorphism defined on $K[\mathrm{ATree}]$ by

$$B_+(T^1 \dots T^n) \mapsto B_+(T^1 \dots T^n) \otimes \emptyset + (\cdot, B_+)\big(\Delta_{\mathrm{CK}}(T^1) \otimes \dots \otimes \Delta_{\mathrm{CK}}(T^n)\big).$$

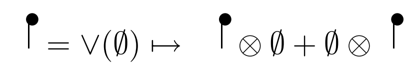
The graded dual of this $\mathcal{C}om$-Hopf algebra is isomorphic (via a graded isomorphism) to a noncommutative cocommutative Hopf algebra on trees introduced by Grossman and Larson [GL89], see [Pan00],[Hof03].

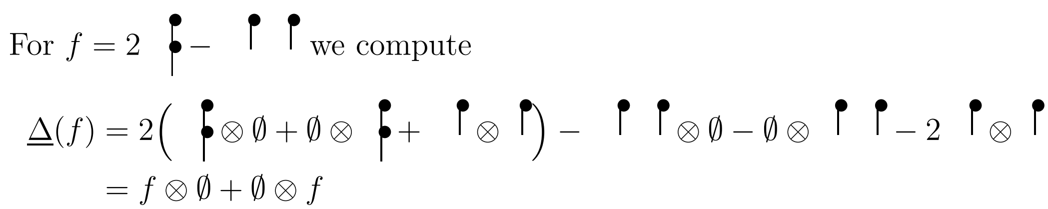
Remark 3.5.4. There is an alternative description of the comultiplication Δ_{CK} and also of $\underline{\Delta}$ using the concept of admissible cuts (see 2.2.11). Since any non-full admissible cut of $T = \vee(T^1 \dots T^n)$ corresponds to n admissible cuts (of $T^1, \dots, T^n$), it is not hard to prove by induction that the image of T under $\underline{\Delta}$ (or Δ_{CK}) is given by

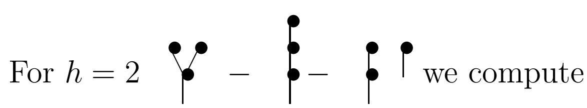
$$\sum_{C \text{admissible cut}} C(T) \otimes R^C(T),$$

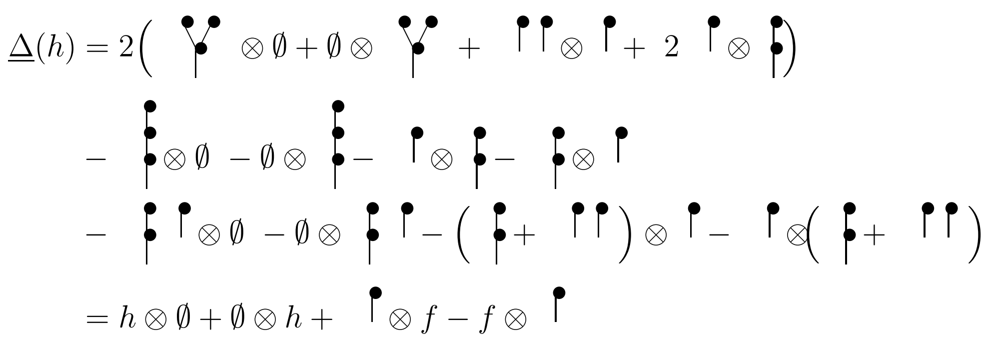
see [CK98].

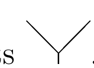
Example 3.5.5. The comultiplication $\underline{\Delta}$ maps









The elements , f, and h are in fact images of the elements , f, and h of Example (3.4.7) under the homomorphism Θ defined in the following.

Definition 3.5.6. Let $\Theta : (K\mathrm{YTree}^{\infty}, *) \to K\langle\mathrm{PTree}\rangle$ be the (graded) algebra homomorphism which is uniquely defined by $\Theta(|) = \emptyset$, $\Theta(\mathbb{V}(Z)) = (\vee \circ \Theta)(Z), Z \in \mathrm{YTree}^{\infty}$.

Proposition 3.5.7. *([Hol03], [Foi])*

The graded algebra homomorphism Θ *provides a Hopf algebra isomorphism between the Hopf algebra* $\big(K\mathrm{YTree}^\infty, *, \Delta_{\mathrm{LR}}\big)$ *of [LR98] and the noncommutative version* $(K\langle\mathrm{PTree}\rangle, \underline{\Delta})$ *of the Hopf algebra of [CK98].*

Proof. Via Θ, $\mathrm{YTree}^{(1)} = \{\mathbb{V}(|)\}$ corresponds to $\mathrm{PTree}_1 = \{\ \bullet\ \}$.

Next, $\mathrm{YTree}^{(2)} = \{\mathbb{V}(\curlyvee)\}$ corresponds to $\mathrm{PTree}_2 = \{\ \bullet\!\!-\!\!\bullet\ \}$.

Let $\xi : (K\langle\mathrm{PTree}\rangle, \cdot) \to (K\mathrm{YTree}^\infty, *)$ be the algebra homomorphism given by $\xi(\emptyset) = |$, $\xi\big(T\big) = \mathbb{V}\big(\xi(\neg(T))\big)$, for planar trees T. Then $\Theta\big(\xi(T)\big) = \big(\vee\circ\Theta\circ\xi\circ\neg\big)(T) = T$ by induction. Similarly $(\xi\circ\Theta)\big(\mathbb{V}(Z)\big) = \mathbb{V}(Z)$, for all Z in YTree^∞.

It is left to show that $(\Theta\otimes\Theta)\circ\Delta = \underline{\Delta}\circ\Theta$. Clearly $(\Theta\otimes\Theta)(\Delta(|)) = \emptyset\otimes\emptyset = \underline{\Delta}(\emptyset)$. Again, we use induction over the degree and assume that $(\Theta\otimes\Theta)(\Delta(S)) = \underline{\Delta}(S)$ for all $S \in K\mathrm{YTree}^\infty$ of degree $< r$.

Let $\mathbb{V}(Z) \in \mathrm{YTree}^\infty$ be a tree of degree r. Thus Z is a tree of degree $r-1$.

$\mathbb{V}(Z)$ is mapped by $(\Theta\otimes\Theta)\circ\Delta$ onto $\Theta(\mathbb{V}(Z))\otimes\emptyset + \big(\Theta\otimes(\Theta\circ\mathbb{V})\big)(\Delta(Z))$

$= \vee(\Theta(Z))\otimes\emptyset + (id,\vee)\big((\Theta\otimes\Theta)\circ\Delta(Z)\big) = \vee(\Theta(Z))\otimes\emptyset + (id,\vee)\big(\underline{\Delta}(\Theta(Z))\big)$ $= \underline{\Delta}\big(\vee(\Theta(Z))\big) = (\underline{\Delta}\circ\Theta)\big(\mathbb{V}(Z)\big)$.

The elements $\mathbb{V}(Z)$ of degree r together with lower degree elements generate all elements of degree r in $K\mathrm{YTree}^\infty$. Thus the equation holds, and Θ is a (graded) Hopf algebra isomorphism.

$\square$

3.6 The Brouder-Frabetti Hopf algebra

The use of a noncommutative $\mathcal{A}s$-Hopf algebra was proposed by C. Brouder and A. Frabetti [BF00] for renormalization. It is also a $\mathcal{D}end$-Hopf algebra, via a nontrivial isomorphism.

The Brouder-Frabetti Hopf algebra is an $\mathcal{A}s$-Hopf algebra structure on the vector space $K\mathrm{YTree}^\infty$, with multiplication map and comultiplication Δ_{BF} both different from the corresponding maps of the Loday-Ronco Hopf algebra. In [BF03] the name $\widetilde{H^\alpha}$ is given to the opposite algebra and coalgebra structure.

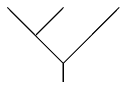
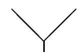
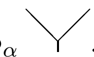
Definition 3.6.1. Let the vector space $K\mathrm{YTree}^\infty$ be provided with the operation $\circ_\alpha$ as an associative multiplication, see Remark (2.3.6).

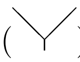
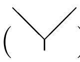
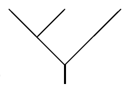
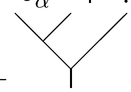
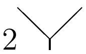
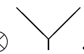
We denote the multiplication $(\circ_\alpha\otimes\circ_\alpha)\circ\tau_2$ on the tensor product $K\mathrm{YTree}^\infty\otimes K\mathrm{YTree}^\infty$ also by $\circ_\alpha$.

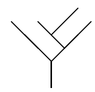
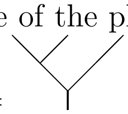
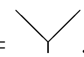
The comultiplication Δ_{BF} defined on $\big(K\mathrm{YTree}^\infty, \circ_\alpha\big)$ is the algebra homomorphism given by $\Delta_{\mathrm{BF}}(|) = |\otimes|$, $\Delta_{\mathrm{BF}}(\curlyvee) = \curlyvee\otimes| + |\otimes\curlyvee$, and, for $T = T^l\vee T^r$ a planar binary tree $\neq |$,

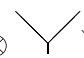
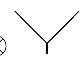
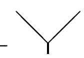
$$\Delta_{\mathrm{BF}}\big(\mathbb{V}(T)\big) = \mathbb{V}\big(T\big)\otimes| + (\mathrm{id}\otimes\mathbb{V})\big((\Delta_{\mathrm{BF}}(\mathbb{V}(T^r)) - \mathbb{V}(T^r)\otimes|)\circ_\alpha\Delta_{\mathrm{BF}}(T^l)\big).$$

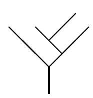
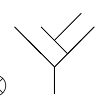
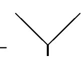
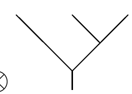
Remark 3.6.2. We note that it suffices to define Δ_{BF} on trees of the form $\mathbb{V}(T) = |\vee T$, because these trees generate $K\mathrm{YTree}^\infty$ with respect to $T \circ_\alpha S$ (recall that this is the grafting of S onto the leftmost leaf of T, see (2.3.6)).

For example,  is the product  $\circ_\alpha$ . Thus Δ_{BF} maps it on

$\Delta_{\mathrm{BF}}($  $) \circ_\alpha \Delta_{\mathrm{BF}}($  $) = | \otimes$  $+$  $\otimes | + 2$  $\otimes$ .

To compute the image of the planar binary tree  under Δ_{BF}, we apply the formula above to $T =$  with $T^l =$ , $T^r = |$ and get

$\mathbb{V}(T) \otimes | + (\mathrm{id} \otimes \mathbb{V})\Big((| \otimes$  $) \circ_\alpha (| \otimes$  $+$  $\otimes |)\Big)$

$=$  $\otimes | + | \otimes$  $+$  $\otimes$ 

Thus Δ_{BF} is not cocommutative.
It is shown in [BF00, BF03] that Δ_{BF} is coassociative.

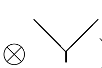
Proposition 3.6.3. *(cf. [Pal02])*
For $T = \vee\!\nearrow\!(T^n \dots T^1)$, $n \geq 1$, $T^j \in \mathrm{YTree}^\infty$, *it holds that*

$$\Delta_{\mathrm{BF}}(\mathbb{V}(T)) = \mathbb{V}(T) \otimes | + \sum \big(T^1_{(1)} \circ_\alpha \dots \circ_\alpha T^n_{(1)}\big) \otimes \mathbb{V}\big(\vee\!\nearrow\!(T^n_{(2)} \dots T^1_{(2)})\big),$$

where $\Delta_{\mathrm{BF}}(S) = \sum S_{(1)} \otimes S_{(2)}$.

Proof. We adapt the argument of [Pal02] to our notation.
If $T = \vee\!\nearrow\!(T^1) = T^1 \vee |$, then by definition, $\Delta_{\mathrm{BF}}\big(\mathbb{V}(T)\big) - \mathbb{V}(T) \otimes |$ is equal to

$$(\mathrm{id} \otimes \mathbb{V})\big((\Delta_{\mathrm{BF}}(\mathbb{V}(|)) - \mathbb{V}(|) \otimes |) \circ_\alpha (\textstyle\sum T^1_{(1)} \otimes T^1_{(2)})\big)$$

$= (\mathrm{id} \otimes \mathbb{V})\big((| \otimes$  $) \circ_\alpha (\sum T^1_{(1)} \otimes T^1_{(2)})\big)$

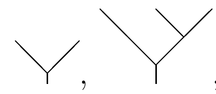
$$= \sum T^1_{(1)} \otimes \mathbb{V}(T^1_{(2)} \vee |).$$

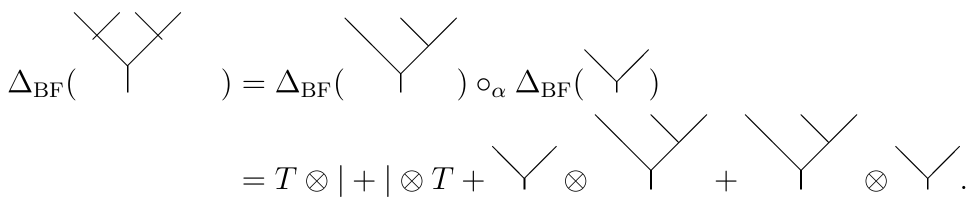
For $T = \vee\!\nearrow\!(T^n \dots T^1)$, since $T^l = T^n$, it holds that $\Delta_{\mathrm{BF}}\big(\mathbb{V}(T)\big) - \mathbb{V}(T) \otimes |$ is equal

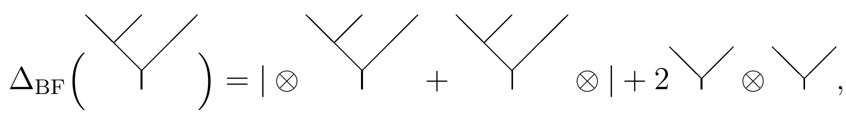
to

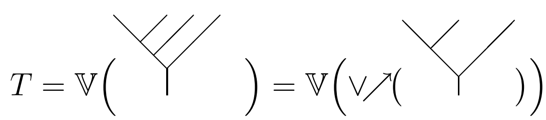
$$
\begin{aligned}
&(\mathrm{id}\otimes\mathbb{V})\Big(\big(\Delta_{\mathrm{BF}}\big(\mathbb{V}(\vee\!\!\nearrow\!(T^{n-1}\dots T^1)\big) - \mathbb{V}(\vee\!\!\nearrow\!(T^{n-1}\dots T^1))\otimes|\big)\circ_\alpha \Delta_{\mathrm{BF}}(T^n)\Big)\\
&= (\mathrm{id}\otimes\mathbb{V})\Big(\big(\sum T^1_{(1)}\circ_\alpha\dots\circ_\alpha T^{n-1}_{(1)}\otimes \mathbb{V}(\vee\!\!\nearrow\!(T^{n-1}_{(2)}\dots T^1_{(2)}))\big)\circ_\alpha \Delta_{\mathrm{BF}}(T^n)\Big)\\
&= (\mathrm{id}\otimes\mathbb{V})\sum T^1_{(1)}\circ_\alpha\dots\circ_\alpha T^{n-1}_{(1)}\circ_\alpha T^n_{(1)}\otimes\big(\mathbb{V}(\vee\!\!\nearrow\!(T^{n-1}_{(2)}\dots T^1_{(2)}))\circ_\alpha T^n_{(2)}\big)\\
&= (\mathrm{id}\otimes\mathbb{V})\sum T^1_{(1)}\circ_\alpha\dots\circ_\alpha T^n_{(1)}\otimes \vee\!\!\nearrow\!(T^n_{(2)}\dots T^1_{(2)})
\end{aligned}
$$

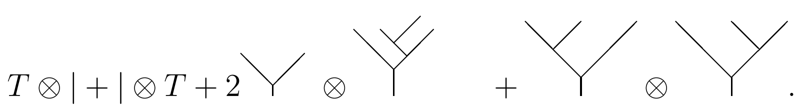
by induction on n. $\square$

Example 3.6.4. It follows immediately that all right combs , . . . , are primitive. Therefore



Since 

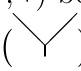
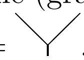
the tree  is mapped by Δ_{BF} onto



Definition 3.6.5. We define, following [Pal02], for $T^1,\dots,T^n$ planar binary trees, the element $\Gamma(T^1,T^2,\dots,T^n)$ of $(K\mathrm{YTree}^\infty,*)$ by

$$
\vee\!\!\nearrow\!(T^1.T^2\dots T^n) - \sum_{j=1}^{n-1}\vee\!\!\nearrow\!\Big(\underbrace{T^1.T^2\dots T^{j-1}.\big(T^j}_{j} * \vee\!\!\nearrow\!(T^{j+1}.T^{j+2}\dots T^n)\big)\Big),
$$

where $\vee\!\!\nearrow$ is applied K-linearly. For $n=1$, we get $\Gamma(T^1)=\vee\!\!\nearrow\!(T^1)=T^1\vee|$.

Let $\Psi : (K\mathrm{YTree}^\infty, \circ_\alpha) \to (K\mathrm{YTree}^\infty, *)$ be the (graded) algebra homomorphism which is uniquely defined by $\Psi(|) = |$, $\Psi($  $) =$ , and

$$\Psi\big(\mathbb{V}(T)\big) = (\mathbb{V} - \vee\!\!\nearrow)\Big(\Gamma(\Psi(T^1), \ldots, \Psi(T^n))\Big) = (\mathbb{V} - \Gamma)\Big(\Gamma(\Psi(T^1), \ldots, \Psi(T^n))\Big)$$

if $T = \vee\!\!\nearrow(T^n \ldots T^1)$ is the right comb presentation of (2.3.9).

Proposition 3.6.6. *The graded algebra homomorphism Ψ provides a Hopf algebra isomorphism between the Hopf algebra $(K\mathrm{YTree}^\infty, \circ_\alpha, \Delta_{\mathrm{BF}})$ of [BF00] and the Hopf algebra $\big(K\mathrm{YTree}^\infty, *, \Delta_{\mathrm{LR}}\big)$ of [LR98].*

Proof. We sketch the proof, following [Pal02].

One checks, using Lemma (3.4.6),

$$\Delta_{\mathrm{LR}}(\Gamma(T^1, \ldots, T^n)) = \Gamma(T^1, \ldots, T^n) \otimes | + \sum T^1_{(1)} * \ldots * T^n_{(1)} \otimes \Gamma(T^1_{(2)}, \ldots, T^n_{(2)}).$$

The application of Δ_{LR} to $\Psi\big(\mathbb{V}(T)\big) = (\mathbb{V} - \vee\!\!\nearrow)\Big(\Gamma(\Psi(T^1), \ldots, \Psi(T^n))\Big)$ and subtraction of the term $\Psi\big(\mathbb{V}(T)\big) \otimes |$ yields, by Lemma (3.4.6),

$$\begin{aligned}
&\sum \Gamma(\Psi(T^1), \ldots, \Psi(T^n))_{(1)} \otimes (\mathbb{V} - \vee\!\!\nearrow)\big(\Gamma(\Psi(T^1), \ldots, \Psi(T^n))_{(2)}\big) \\
&= \sum (\Psi(T^1))_{(1)} * \ldots * (\Psi(T^n))_{(1)} \otimes (\mathbb{V} - \vee\!\!\nearrow)\Gamma(\Psi(T^1)_{(2)}, \ldots, \Psi(T^n)_{(2)}).
\end{aligned}$$

By induction, we can assume $(\Psi(T^j))_{(1)} = \Psi(T^j_{(1)})$ and $(\Psi(T^j))_{(2)} = \Psi(T^j_{(2)})$. Thus $\Delta_{\mathrm{LR}}\big(\Psi\mathbb{V}(T)\big)$ is given by

$$\Psi\mathbb{V}(T) \otimes | + \sum (\Psi(T^1_{(1)}) * \ldots * \Psi(T^n_{(1)})) \otimes (\mathbb{V} - \vee\!\!\nearrow)(\Gamma(\Psi(T^1_{(2)}), \ldots, \Psi(T^n_{(2)}))$$

On the other hand, in view of Proposition (3.6.3), we get that $\Psi \otimes \Psi\big(\Delta_{\mathrm{BF}}(\mathbb{V}(T))\big)$ is given by

$$\begin{aligned}
&\Psi\mathbb{V}(T) \otimes | + \sum \Psi\big(T^1_{(1)} \circ_\alpha \ldots \circ_\alpha T^n_{(1)}\big) \otimes \Psi\mathbb{V}\big(\vee\!\!\nearrow(T^n_{(2)} \ldots T^1_{(2)})\big) \\
&= \Psi\mathbb{V}(T) \otimes | + \sum \big(\Psi T^1_{(1)} * \ldots * \Psi T^n_{(1)}\big) \otimes (\mathbb{V} - \vee\!\!\nearrow)(\Gamma(\Psi(T^1_{(2)}), \ldots, \Psi(T^n_{(2)})).
\end{aligned}$$

Thus Ψ is a Hopf algebra homomorphism. One then checks that Ψ is a graded bijection. $\square$

Remark 3.6.7. The construction of a Hopf algebra isomorphism from the Brouder-Frabetti Hopf algebra $\big(K\mathrm{YTree}^\infty, \circ_\alpha, \Delta_{\mathrm{BF}}\big)$ to the graded dual of $(K\langle\mathrm{PTree}\rangle, \underline{\Delta})$ is given in [Foi] (see also [Foi02]). There it is also shown that the Hopf algebra $\big(K\langle\mathrm{PTree}\rangle, \underline{\Delta}\big)$ is self-dual, and thus also $\big(K\mathrm{YTree}^\infty, \circ_\alpha, \Delta_{\mathrm{BF}}\big)$, $\big(K\mathrm{YTree}^\infty, *, \Delta_{\mathrm{LR}}\big)$.

For the Loday-Ronco Hopf algebra, Aguiar and Sottile ([AS04]) construct a different basis $\{M_T : T \in \mathrm{YTree}^\infty\}$, called the monomial basis. They give a different construction for an isomorphism between $\big(K\langle\mathrm{PTree}\rangle, \underline{\Delta}\big)$ and $\big(\mathrm{YTree}^\infty, \Delta_{\mathrm{LR}}\big)$.

3.7 Prim$\mathcal{P}$ and the Milnor-Moore Theorem

Let $\mathcal{P}$ be equipped with a coherent unit action, and let V_X be the vector space with basis $X = \{x_1, x_2, \ldots\}$. The co-addition Δ_a provides the free $\mathcal{P}$-algebra $F_{\mathcal{P}}(V_X)$ with the structure of a strictly graded $\mathcal{P}$-Hopf algebra, see Example (3.3.6). There exists an operad Prim$\mathcal{P}$, as the following Lemma holds.

Lemma 3.7.1. *(cf. [GH03],[Lod03b])*
The vector space $\mathrm{Prim}(F_{\mathcal{P}}(V_X))$ *is closed under insertion: Every $\mathcal{P}$-algebra homomorphism $\eta_{(g_1,g_2,\ldots)}$ given by $x_i \mapsto g_i$, for $g_i \in \mathrm{Prim}(F_{\mathcal{P}}(V_X))$, maps primitive elements on primitive elements.*

Proof. Since $g_i \in \mathrm{Prim}(F_{\mathcal{P}}(V_X))$ is equivalent to the compatibility of $\eta_{(g_1,g_2,\ldots)}$ with Δ_a, the map $\eta_{(g_1,g_2,\ldots)}$ is a $\mathcal{P}$-Hopf algebra homomorphism, and the assertion follows. □

Definition 3.7.2. (cf.[Lod03b])
Let $\mathcal{P}$ be equipped with a coherent unit action. Let Δ_a be the co-addition on $F_{\mathcal{P}}(V_X)$, $X = \{x_1, x_2, \ldots\}$.

Then we define Prim$\mathcal{P}$ to be the operad with free algebra functor Prim$F_{\mathcal{P}}$ and composition maps induced by insertion.

Example 3.7.3. It holds that $\mathrm{Prim}\mathcal{C}om = \mathrm{Vect}_K$. The operad Prim$\mathcal{A}s$ is $\mathcal{L}ie$.
This follows from the Theorem of Friedrichs (cf. [Reu]), which states that Lie polynomials are exactly the polynomials in non-commuting associative variables which are primitive under Δ_a.

Remark 3.7.4. If A is a cocommutative connected graded $\mathcal{A}s$-Hopf algebra (over a characteristic 0 field K), then

$$A \cong U(\mathrm{Prim}(A))$$

is isomorphic to the universal enveloping bialgebra of its space of primitive elements. This is the theorem of Milnor-Moore ([MM65], see also [Qui]).

Together with the theorem of Poincaré-Birkhoff-Witt, it follows that the category of such Hopf algebras is equivalent to the category of Lie algebras.

An analogon of the Milnor-Moore theorem has been proven by Ronco [Ron02] for $\mathcal{D}end$-Hopf algebras (compare also [Cha02]).

The role of $\mathcal{L}ie$-algebras is played by $\mathcal{B}race$-algebras (defined below), special pre-Lie algebras equipped with n-ary operations $\langle \ldots \rangle : A^{\otimes n} \to A$ for each n.

The pre-Lie bracket $\langle\ ,\ \rangle : \mathrm{Prim}(F_{\mathcal{D}end}V)^{\otimes 2} \to \mathrm{Prim}(F_{\mathcal{D}end}V)$ is given by

$$\langle f, g\rangle = f \prec g - g \succ f.$$

Ronco's results show that the primitive elements of the Loday-Ronco Hopf algebra, the noncommutative planar Connes-Kreimer Hopf algebra, and the Hopf algebra of Brouder-Frabetti, are a free brace algebra on one generator. It follows that $\mathrm{Prim}\mathcal{D}end = \mathcal{B}race$ (cf. also [Lod03b]).

For different constructions of the space of primitive elements see also [Foi], [AS02], and [HNT04].

In [LR04] Loday and Ronco prove a Milnor-Moore and Poincaré-Birkhoff-Witt theorem for Hopf algebras H that are equipped with two multiplications $*$ and $\cdot$, such that the comultiplication Δ is a homomorphism with respect to $*$, and such that $(H, \cdot, \Delta)$ is unital infinitesimal:

$$\Delta(x \cdot y) = (x \otimes 1) \cdot \Delta(y) + \Delta(x) \cdot (1 \otimes y) - x \otimes y$$

They show that (non-dg) B_∞-algebras occur as the primitive elements.

We sketch the definition of (non-dg) B_∞-algebras below, following [LR04].

Definition 3.7.5. (cf. [LR03],[Ron00a],[Ron02])

Let A be a K-vector space together with a family

$$M_{pq} : A^{\otimes p} \otimes A^{\otimes q} \to A, \quad p \geq 0, q \geq 0$$

of $(p+q)$-ary operations be given, with

$$M_{00} = 0,\ M_{10} = \mathrm{id}_A = M_{01},\ \text{and } M_{n0} = 0 = M_{0n} \text{ for } n \geq 2.$$

For all $k \geq 0$, and all $i = (i_1, i_2, \ldots, i_k), j = (j_1, j_2, \ldots, j_k)$, one denotes by

$$M_{i_1 j_1} M_{i_2 j_2} \ldots M_{i_k j_k} : A^{\otimes p} \otimes A^{\otimes q} \to A^{\otimes k},\ \text{with } p := i_1 + \ldots + i_k, q := j_1 + \ldots + j_k,$$

the map which sends $u_1.u_2 \ldots u_p \otimes v_1.v_2 \ldots v_q$ to

$$M_{i_1 j_1}(u_1 \ldots u_{i_1} \otimes v_1 \ldots v_{j_1}).M_{i_2 j_2}(\ldots u_{i_1+i_2} \otimes \ldots v_{j_1+j_2}) \ldots M_{i_k j_k}(\ldots u_p \otimes \ldots v_q).$$

Let $* : A^{\otimes p} \otimes A^{\otimes q} \to A$ be given by

$$u_1.u_2 \ldots u_p * v_1.v_2 \ldots v_q = \sum_{k \geq 1} \sum_{i,j} M_{i_1 j_1} \ldots M_{i_k j_k}(u_1.u_2 \ldots u_p \otimes v_1.v_2 \ldots v_q),$$

where the sum is over all k-tuples i, j with $p = i_1 + \ldots + i_k, q = j_1 + \ldots + j_k$.

Then

$$\begin{aligned} u * v &= M_{11}(u \otimes v) + u.v + v.u \\ u.v * w &= M_{21}(u.v \otimes w) + (u * w).v + u.(v * w) - u.w.v \\ u * v.w &= M_{12}(u \otimes v.w) + (u * v).w + v.(u * w) - v.u.w. \end{aligned}$$

The space A together with the operations M_{pq} is called a B_∞-algebra, iff for every triple (i, j, k) of positive integers the relation

$$(u_1.u_2 \ldots u_i * v_1.v_2 \ldots v_j) * w_1.w_2 \ldots w_k = u_1.u_2 \ldots u_i * (v_1.v_2 \ldots v_j * w_1.w_2 \ldots w_k)$$

holds.

For $(i,j,k) = (1,1,1)$ this is

$$\begin{aligned} M_{11}\big(u \otimes M_{11}(v \otimes w)\big) &+ M_{12}\big(u \otimes (v.w + w.v)\big) \\ &= M_{21}\big((u.v + v.u) \otimes w\big) + M_{11}\big(M_{11}(u \otimes v) \otimes w\big). \end{aligned}$$

If $M_{pq} = 0$ for all (p,q) with $p \geq 2$, then A is called a brace algebra, and one denotes the n-ary operations $M_{1,n-1}$ for $n \geq 2$ by

$$\langle \ldots \rangle : A^{\otimes n} \to A.$$

Then, e.g. the relation for $(i,j,k) = (1,1,1)$, reads

$$\langle \langle u.v \rangle .w \rangle = \langle u. \langle v.w \rangle \rangle + \langle u.v.w \rangle + \langle u.w.v \rangle.$$

Interchanging v and w one gets the pre-Lie relation

$$\langle \langle u.v \rangle .w \rangle - \langle u. \langle v.w \rangle \rangle = \langle \langle u.w \rangle .v \rangle - \langle u. \langle w.v \rangle \rangle,$$

cf. [CL01], [Gers63].

Chapter 4

Primitive elements of $\mathcal{M}ag$- and $\mathcal{M}ag_\omega$-algebras

4.1 The free $\mathcal{M}ag$- and $\mathcal{M}ag_\omega$-algebras

Let K be a field of characteristic 0, and let $X = \{x_1, x_2, \ldots\}$ be a finite or countable set of variables. Let V_X be the vector space with basis X.

Remark 4.1.1. We recall, see Example (2.4.6), that the n-th space of the Stasheff operad $\mathcal{M}ag_\omega$, i.e. the space $\Gamma(M)(n)$ of the free operad $\Gamma(M)$ generated by operations $\vee^k \in M_k, k \geq 2$, is given by linear combinations of reduced (non-labeled) planar trees with n leaves. The operad structure is given by insertion, see (2.4.6).

Super-Catalan numbers are the coefficients of the generating series of $\mathcal{M}ag_\omega$:

$$f^{\mathcal{M}ag_\omega}(t) = \sum_{n\geq 1} \frac{\dim \mathcal{M}ag_\omega(n)}{n!} t^n = \sum_{n\geq 1} C_n t^n = \frac{1}{4}(1 + t - \sqrt{1 - 6t + t^2}).$$

For any k, the operation $\vee^k$ is given by the grafting operation $\vee$ restricted to planar forests consisting of k trees.

Let $M^X = (M_k^X)_{k\geq 0}$ denote the sequence

$$M_0^X = X, \quad M_1^X = \emptyset, \quad M_k^X = M_k = \{\vee^k\} \text{ (for } k \geq 2).$$

Then the free $\mathcal{M}ag_\omega$-algebra

$$F_{\mathcal{M}ag_\omega}(V_X) = \bigoplus_{n=1}^{\infty} F_{\mathcal{M}ag_\omega}(V_X)^{(n)} = \bigoplus_{n=1}^{\infty} \mathcal{M}ag_\omega(n) \otimes_{\Sigma_n} \left(V_X\right)^{\otimes n}$$

has the set $\mathrm{PRTree}\{M^X\}$ of reduced planar trees with leaves labeled by X as a vector space basis. It is naturally graded, such that the planar trees with n leaves form a basis of $F_{\mathcal{M}ag_\omega}(V_X)^{(n)}$.

Lemma 4.1.2. *The operad $\mathcal{M}ag_\omega$ is equipped with a (unique) unit action which respects the operations $\vee^k$, i.e.*

$$\begin{aligned} &\vee^k \circ_{k,i} (\eta) = \vee^{k-1}, \ \ \textit{all } i \\ &\vee^2 \circ_{2,1} (\eta) = \vee^2 \circ_{2,2} (\eta) = \mathrm{id}\,. \end{aligned}$$

This unit action is coherent.

Proof. Since $\mathcal{M}ag_\omega$ is freely generated by the operations $\vee^k$, we can define the unit action by the formulas above, see Definition (3.2.3). The unit action is compatible with the (empty) set of relations of $\mathcal{M}ag_\omega$, and coherence also follows.

□

Definition 4.1.3. We denote the free $\mathcal{M}ag_\omega$-algebra with unit 1 by

$$\begin{aligned} &K\{X\}_\infty = \bigoplus_{n=0}^{\infty} K\{X\}_\infty^{(n)}, \\ &K\{X\}_\infty^{(0)} = K1, \quad K\{X\}_\infty^{(n)} = F_{\mathcal{M}ag_\omega}(V_X)^{(n)} \ (n \geq 1). \end{aligned}$$

We set $\vee^k(1, 1, \ldots, 1) = 1$ (all k).

We identify 1 with the empty tree $\emptyset$ and call

$$\mathrm{PRTree}'\{M^X\} = \mathrm{PRTree}\{M^X\} \cup \{\emptyset\}$$

the set of monomials of $K\{X\}_\infty$.

Remark 4.1.4. The binary quadratic operad $\mathcal{M}ag$ with generating series

$$f^{\mathcal{M}ag}(t) = \sum_{n\geq 1} \frac{\dim \mathcal{M}ag(n)}{n!} t^n = \sum_{n \geq 1} c_n t^n = \frac{1 - \sqrt{1-4t}}{2}$$

is contained as a sub-operad in $\mathcal{M}ag_\omega$, i.e. there is an inclusion of operads

$$\mathcal{M}ag \to \mathcal{M}ag_\omega.$$

(The vector subspace given by planar binary trees with labeled leaves is closed under the operation $\vee^2$.)

The unit action of $\mathcal{M}ag_\omega$ extends the canonical unit action on $\mathcal{M}ag$ induced by $1 \cdot a = a \cdot 1 = a$ (for every $\mathcal{M}ag$-algebra A, every $a \in A$), see (3.2.2).

The free $\mathcal{M}ag$-algebra with unit 1 can be identified with the space of labeled binary trees $K\{X\} \subset K\{X\}_\infty$, equipped with the free binary operation $\cdot = \vee^2$.

We pass freely from planar binary trees to parenthesized strings, see (2.2.3). We call a planar binary tree with labeled leaves a monomial of $K\{X\}$, and we call the elements of $K\{X\}$ (and $K\{X\}_\infty$) polynomials.

We can embed $K\{X\}_\infty$ into its completion $K\{\{X\}\}_\infty = \prod_{n=0}^{\infty} K\{X\}_\infty^{(n)}$, the free complete $\mathcal{M}ag_\omega$-algebra generated by X. Similarly defined is the free $\mathcal{M}ag$-power series algebra $K\{\{X\}\}$.

We have the canonical degree and order functions.

The degree of a tree $T \in K\{X\}_\infty^{(n)}$ is n, i.e. the degree of T is the number of its leaves.

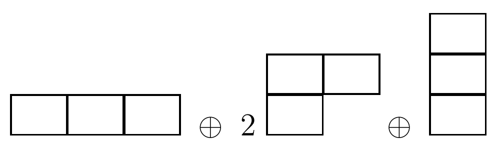
The multi-degree of a tree $T \in K\{X\}_\infty$ is $\mathbf{n} = (n_1, n_2, \ldots)$, if n_j is the number of leaves with label x_j for all $j \in \mathbb{N}$.

Remark 4.1.5. While $\mathcal{A}s(n) = K\Sigma_n$ corresponds to the regular representation, the Σ_n-module $\mathcal{M}ag(n)$ is given by c_n copies of $K\Sigma_n$, and $\mathcal{M}ag_\omega(n)$ is given by C_n copies of $K\Sigma_n$.

The associated GL-modules describe the spaces $K\langle X\rangle^{(n)}$, $K\{X\}^{(n)}$, and $K\{X\}_\infty^{(n)}$.

The first degree where $K\{X\}^{(n)}$ and $K\{X\}_\infty^{(n)}$ differ is degree 3.

Here the regular representation



occurs in two copies for $K\{X\}^{(3)}$, and in three copies for $K\{X\}_\infty^{(3)}$. The extra copy is induced by the 3-corolla.

For $X = \{x_1, x_2, \ldots, x_m\}$, the Hilbert series $\mathrm{Hilb}(A, t_1, \ldots, t_m)$ of the multigraded vector spaces $A = K\{X\}_\infty$ and $A = K\{X\}$ is defined by

$$\mathrm{Hilb}(A, t_1, \ldots, t_m) = \sum_{\mathbf{n} \in \mathbb{N}^m} \dim A^{(n_1, n_2, \ldots, n_m)} t_1^{n_1} t_2^{n_2} \cdots t_m^{n_m}$$

where the vector space $A^{(n_1, n_2, \ldots, n_m)}$ is the space of homogeneous elements having multi-degree $(n_1, n_2, \ldots, n_m)$.

Again, the coefficients for $K\{X\}_\infty$ and $K\{X\}$ are just given by the coefficients for $K\langle X\rangle$, multiplied by C_n or c_n.

4.2 Partial derivatives on $\mathcal{M}ag_\omega$-algebras

Definition 4.2.1. A derivation D on a $\mathcal{M}ag$-algebra A is a K-linear map which fulfills the Leibniz rule

$$D(u \cdot v) = D(u) \cdot v + u \cdot D(v) \text{ for all } u, v \in A.$$

A derivation D on a $\mathcal{M}ag_\omega$-algebra A is a K-linear map which fulfills

$$D(\vee^n(v_1, \ldots, v_n)) = \sum_{i=1}^{n} \vee^n(v_1, \ldots, v_{i-1}, D(v_i), v_{i+1}, \ldots, v_n) \text{ for all } v_i \in A.$$

Lemma 4.2.2. *Every mapping $\delta : X \to K\{X\}$, or $X \to K\{X\}_\infty$ respectively, can be extended to a unique derivation of $K\{X\}$, $K\{X\}_\infty$ respectively. Here $\delta|_K = 0$.*

Proof. As usual, one defines $\delta(u \cdot v) = \delta(u) \cdot v + u \cdot \delta(v)$, or $\delta\big(\vee^n(v_1, \ldots, v_n)\big) = \sum_{i=1}^n \vee^n(v_1, \ldots, v_{i-1}, \delta(v_i), v_{i+1}, \ldots, v_n)$ for $u, v, v_i \in X$. $\square$

Definition 4.2.3. Let $\partial_k : K\{X\}_\infty \to K\{X\}_\infty$ be the derivation given by

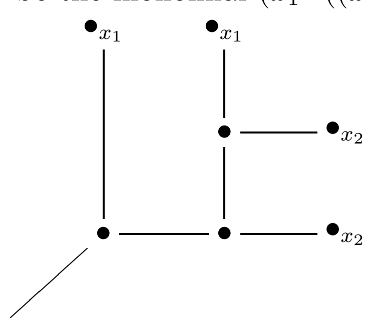
$$\partial_k(x_l) := \begin{cases} 1 & : \ k = l \\ 0 & : \ k \neq l \end{cases}$$

Let $\partial_{kj} : K\{X\}_\infty \to K\{X\}_\infty$ be the derivation given by

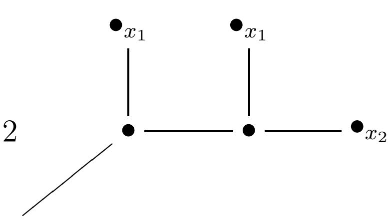
$$\partial_{kj}(x_l) := \begin{cases} x_j & : \ k = l \\ 0 & : \ k \neq l \end{cases}$$

The restrictions on $K\{X\}$ are also denoted by $\partial_k, \partial_{kj}$.

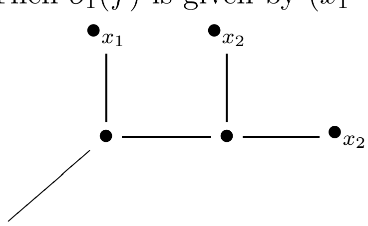
Example 4.2.4. Let f be the monomial $(x_1 \cdot ((x_1 \cdot x_2) \cdot x_2))$, which corresponds to the planar binary tree



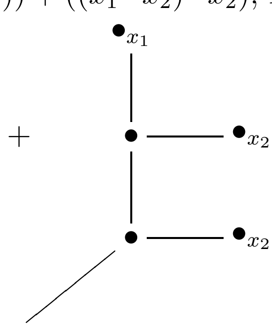
Then $\partial_2(f)$ is given by $2(x_1 \cdot (x_1 \cdot x_2))$ or

2 

Then $\partial_1(f)$ is given by $(x_1 \cdot (x_2 \cdot x_2)) + ((x_1 \cdot x_2) \cdot x_2)$, i.e.



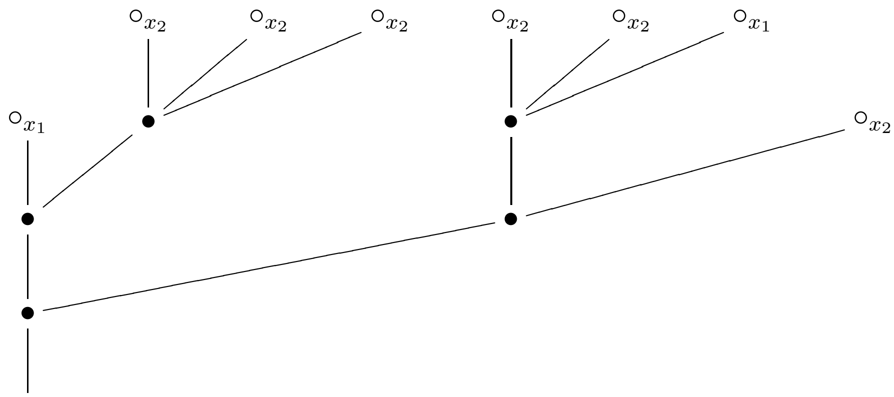
$+$



Example 4.2.5. If f is the monomial

$$\vee^2\Big(\vee^2\big(x_1, \vee^3(x_2, x_2, x_2)\big)\big), \vee^2\big(\vee^3(x_2, x_2, x_1), x_2\big)\Big),$$

which corresponds to



then $\partial_{12}(f)$ is given by

$$\begin{aligned} &\vee^2\ \Big(\vee^2\big(x_2, \vee^3(x_2, x_2, x_2)\big)\big), \vee^2\big(\vee^3(x_2, x_2, x_1), x_2\big)\Big) \\ &+ \vee^2\Big(\vee^2\big(x_1, \vee^3(x_2, x_2, x_2)\big)\big), \vee^2\big(\vee^3(x_2, x_2, x_2), x_2\big)\Big). \end{aligned}$$

and $\partial_1(f)$ is given by

$$\begin{aligned} &\vee^2\ \Big(\vee^3(x_2, x_2, x_2)\big), \vee^2\big(\vee^3(x_2, x_2, x_1), x_2\big)\Big) \\ &+ \vee^2\Big(\vee^2\big(x_1, \vee^3(x_2, x_2, x_2)\big)\big), \vee^2\big(\vee^2(x_2, x_2), x_2\big)\Big). \end{aligned}$$

Proposition 4.2.6. *Let $T \in K\{X\}_\infty$ be a monomial, i.e. an element of the set* $\mathrm{PRTree}\{M^X\}$, $x_k \in X$.

Let $I_k \subseteq \mathrm{Le}(T)$ be the subset given by all leaves with label x_k.

Then, for any j, $\partial_{kj}(T)$ is given by

$$\partial_{kj}(T) = \sum_{\nu \in I_k} T_{\nu \to x_j}$$

where $T_{\nu \to x_j}$ denotes the tree which is obtained from T by labeling the leaf ν by x_j (instead by x_k).

If, for $\nu \in I_k$, $\{\nu\}^c$ denotes the complement $\mathrm{Le}(T) - \{\nu\}$ *of ν in the set of leaves of T, then $\partial_k(T)$ is given by*

$$\partial_k(T) = \sum_{\nu \in I_k} \mathrm{red}(T|\{\nu\}^c)$$

where $\mathrm{red}(T|\{\nu\}^c)$ *is the reduction of the leaf-restriction of T onto $\{\nu\}^c$, see (2.2.5) and (2.2.15).*

4.3 Co-addition $\mathcal{M}ag$- and $\mathcal{M}ag_\omega$-Hopf algebras

Since the unit actions for $\mathcal{P} \in \{\mathcal{M}ag, \mathcal{M}ag_\omega\}$ are coherent, it makes sense to look at the $\mathcal{P}$-algebras $K\{X\}_\infty \otimes K\{X\}_\infty$ and $K\{X\} \otimes K\{X\}$.

Both $K\{X\}$ and $K\{X\}_\infty$ are naturally graded, $K\{X\}_\infty = \bigoplus_{n\in\mathbb{N}} K\{X\}_\infty^{(n)}$, $K\{X\} = \bigoplus_{n\in\mathbb{N}} K\{X\}^{(n)}$. Homogeneous elements of degree n are the K-linear combinations of trees with n leaves. The operations $\vee^n$ clearly respect this grading.

Definition 4.3.1. Let $\Delta_a : K\{X\}_\infty \to K\{X\}_\infty \otimes K\{X\}_\infty$ be the $\mathcal{M}ag_\omega$-algebra homomorphism defined by

$$x_i \mapsto x_i \otimes 1 + 1 \otimes x_i, \text{ for all } x_i \in X.$$

The map Δ_a is called the co-addition.

Remark 4.3.2. We note that Δ_a is cocommutative. The notion of cocommutativity is defined for $\mathcal{M}ag_\omega$ and $\mathcal{M}ag$, see Definition (3.3.7).

The restriction $\Delta_a|_{K\{X\}}$ is a $\mathcal{M}ag$-algebra homomorphism

$$\Delta_a : K\{X\} \to K\{X\} \otimes K\{X\},$$

which is also cocommutative.

For example, to compute the image of $(x_1 \cdot x_2) \cdot x_3$ under Δ_a, one computes

$$\begin{aligned}
\Delta_a(x_1 \cdot x_2) &= (x_1 \otimes 1 + 1 \otimes x_1) \cdot (x_2 \otimes 1 + 1 \otimes x_2) \\
&= x_1 \cdot x_2 \otimes 1 + 1 \otimes x_1 \cdot x_2 + x_1 \otimes x_2 + x_2 \otimes x_1 \\
\Delta_a\big((x_1 \cdot x_2) \cdot x_3\big) &= (x_1 \cdot x_2 \otimes 1 + 1 \otimes x_1 \cdot x_2 + x_1 \otimes x_2 + x_2 \otimes x_1) \cdot (x_3 \otimes 1 + 1 \otimes x_3) \\
&= (x_1 \cdot x_2) \cdot x_3 \otimes 1 + 1 \otimes (x_1 \cdot x_2) \cdot x_3 + x_1 \cdot x_2 \otimes x_3 + x_1 \otimes x_2 \cdot x_3 \\
&\quad + x_2 \otimes x_1 \cdot x_3 + x_3 \otimes x_1 \cdot x_2 + x_2 \cdot x_3 \otimes x_1 + x_1 \cdot x_3 \otimes x_2.
\end{aligned}$$

Proposition 4.3.3. *The free unitary $\mathcal{M}ag_\omega$-algebra $K\{X\}_\infty$ on X together with Δ_a is a strictly graded $\mathcal{M}ag_\omega$-Hopf algebra.*

The free unitary $\mathcal{M}ag$-algebra $K\{X\} \subseteq K\{X\}_\infty$ is a strictly graded $\mathcal{M}ag$-Hopf algebra.

Proof. We recall from (3.3.6) that $(\Delta_a \otimes \mathrm{id})\Delta_a = (\mathrm{id} \otimes \Delta_a)\Delta_a$, because both homomorphisms map x_i on $x_i \otimes 1 \otimes 1 + 1 \otimes x_i \otimes 1 + 1 \otimes 1 \otimes x_i$. Here we note that the map Δ_a is (by definition) a $\mathcal{M}ag_\omega$-algebra homomorphism.

It follows that

$$\Delta_a'\big(K\{X\}_\infty^{(n)}\big) \subseteq \sum_{i=1}^{n-1} K\{X\}_\infty^{(i)} \otimes K\{X\}_\infty^{(n-i)}$$

using the $\mathcal{M}ag_\omega$-structure of the tensor product together with the fact that the grading is respected by the operations $\vee^n$.

For the case of $\mathcal{M}ag$, the analogous statements hold.

□

Remark 4.3.4. Consider the $\mathcal{M}ag$-Hopf algebra $(K\{X\},\Delta_a)$. There is a unique K-linear map σ_l, the left-antipode, such that $\sigma_l(w)+w+\sum'\sigma_l(w_{(1)})w_{(2)}=0$, $\deg w\geq 1$, where $\Delta_a(w)=w\otimes 1+1\otimes w+\sum' w_{(1)}\otimes w_{(2)}$, see Remark (3.3.5).

For primitive f, $\sigma_l(f)=-f$. But σ_l is not the anti-homomorphism induced by $x_i\mapsto -x_i$.

Let $\bar{t}$ denote the involution induced by $\overline{t_1t_2}=\bar{t}_2\bar{t}_1$, compare Remark (2.1.15).

Then $\sigma_r:=\bar{\sigma}_l$ given by $\bar{\sigma}_l(\bar{t})=\overline{\sigma_l(t)}$ fulfills $\bar{\sigma}_l(w)+w+\sum w_{(1)}\bar{\sigma}_l(w_{(2)})=0$. Therefore σ_r is the right-antipode. Left and right-antipode do not coincide.

For example, let $X=\{x\}$ consist of one element. Then $\sigma_l(x)=-x$, $\sigma_l(x\cdot x)=x\cdot x$, and we recursively compute that $\sigma_l(x\cdot(x\cdot x))=2x\cdot(x\cdot x)-3(x\cdot x)\cdot x$, $\sigma_l((x\cdot x)\cdot x)=3x\cdot(x\cdot x)-4(x\cdot x)\cdot x$.

On the other hand $\sigma_r(x\cdot(x\cdot x))=3(x\cdot x)\cdot x-4x\cdot(x\cdot x)$, $\sigma_r((x\cdot x)\cdot x)=2(x\cdot x)\cdot x-3x\cdot(x\cdot x)$.

The order of σ_l (and also the order of σ_r) is infinite.

Proposition 4.3.5. *Let $T\in K\{X\}_\infty$ be a monomial.*

For $I\subseteq \mathrm{Le}(T)$, let $\big(\mathrm{red}(T|I),\mathrm{red}(T|I^c)\big)$ be the leaf-split induced by (I,I^c), see (2.2.15).

Then
$$\Delta_a(T)=\sum_{I\subseteq \mathrm{Le}(T)}\mathrm{red}(T|I)\otimes\mathrm{red}(T|I^c).$$

The same formula holds for the restriction of Δ_a to $K\{X\}$, and $T\in K\{X\}$ a monomial.

Proof. For $T=1$, $\mathrm{Le}(T)=\emptyset=I$ and the formula says that $\Delta_a(1)=1\otimes 1$. For $T=x_i\in X$,
$$\Delta_a(T)=T\otimes 1+1\otimes T=\mathrm{red}(T|\mathrm{Le}(T))\otimes\mathrm{red}(T|\emptyset)+\mathrm{red}(T|\emptyset)\otimes\mathrm{red}(T|\mathrm{Le}(T))$$

Else $T=\vee^n(T^1\dots T^n)$, $n\geq 2$. Since Δ_a is by definition a $\mathcal{M}ag_\omega$-algebra homomorphism,
$$\Delta_a(T)=\vee^n\big(\Delta_a(T^1)\dots\Delta_a(T^n)\big).$$
After an iterated use of the $\mathcal{M}ag_\omega$-algebra homomorphism property, we arrive at a sum that has one summand for every subset $I\subseteq\mathrm{Le}(T)$. Here the choice between letting the j-th leaf belong to I or its complement I^c corresponds to the choice of a factor $x_{\nu(j)}\otimes 1$ instead of a factor $1\otimes x_{\nu(j)}$, where $x_{\nu(j)}$ is the label of the j-th leaf of T.

By construction, the reduced leaf-restriction $\mathrm{red}(T|I)$ is the first tensor component of the summand given by I in $\Delta_a(T)$. Analogously, $\mathrm{red}(T|I^c)$ is the second tensor component.

The same proof holds, when $K\{X\}_\infty$ is replaced by $K\{X\}$ and $\mathcal{M}ag_\omega$ is replaced by $\mathcal{M}ag$.

∎

Definition 4.3.6. For any monomial T in $K\{X\}_\infty$ we define a K-linear map $\partial_T : K\{X\}_\infty \to K\{X\}_\infty$ by

$$\Delta_a(f) = \sum_{T \in \mathrm{PRTree}\{M^X\}} T \otimes \partial_T(f),$$

where $f \in K\{X\}_\infty$. Especially, for $T = 1_{K\{X\}_\infty} = \emptyset$, $\partial_\emptyset = \mathrm{id}$.

More generally, for g homogeneous, $g = \sum a_i T^i, a_i \in K$, we also define $\partial_g(f) := \sum a_i \partial_{T^i}(f)$.

We analogously define $\partial_T(f)$ and $\partial_g(f)$ for the binary case, where f, g (or T) are elements of $K\{X\}$.

We call ∂_T a generalized differential operator. It has the following properties.

Proposition 4.3.7. *(i) For all monomials S, T in $K\{X\}_\infty$,*

$$\partial_S \circ \partial_T = \partial_T \circ \partial_S$$

(ii) For any $x_k \in X$, ∂_{x_k} is the derivation ∂_k.

(iii) For T a monomial, and $f_1, \dots, f_p \in K\{X\}_\infty$,

$$\partial_T(\vee^p(f_1 \dots f_p)) = \sum_{\vee^p(T^1.T^2 \dots T^p) = T} \vee^p\big(\partial_{T^1}(f_1) \dots \partial_{T^p}(f_p)\big),$$

where the sum is over all not necessarily non-empty trees $T^1, T^2, \dots, T^p$ such that $\vee^p(T^1.T^2 \dots T^p) = T$.
Especially

$$\partial_T(f_1 \cdot f_2) = f_1 \cdot \partial_T(f_2) + \partial_{T^1}(f_1) \cdot \partial_{T^2}(f_2) + \partial_T(f_1) \cdot f_2$$

if $T = \vee^2(T^1.T^2)$ and $f_1, f_2 \in K\{X\}$ are binary.

Proof. 1) Since Δ_a is cocommutative, we get the formula

$$\begin{aligned}\Delta_a{}^2(f) &= \sum_{S,T \in \mathrm{PRTree}\{M^X\}} S \otimes T \otimes \partial_S \partial_T(f) \\ &= \sum_{S,T \in \mathrm{PRTree}\{M^X\}} S \otimes T \otimes \partial_T \partial_S(f).\end{aligned}$$

Hence (i) holds.

2) In the sum

$$\Delta_a(T) = \sum_{I \subseteq \mathrm{Le}(T)} \mathrm{red}(T|I) \otimes \mathrm{red}(T|I^c).$$

of (4.3.5), $\mathrm{red}(T|I) = x_k$ exactly when $I = \{\nu\}$, $\nu \in I_k$. Here $I_k \subseteq \mathrm{Le}(T)$ is the subset given by all leaves with label x_k.
Since $\partial_k(T)$ is given by

$$\sum_{\nu \in I_k} \mathrm{red}(T|\{\nu\}^c),$$

see Proposition (4.2.6), assertion (ii) follows.

3) Since the co-addition Δ_a is an algebra homomorphism with respect to the operations $\vee^n$, we get (iii).

□

Corollary 4.3.8. *Let S, T in $K\{X\}_\infty$ be monomials, and let $\mu_S(T)$ be the number of subsets I of $\mathrm{Le}(T)$ which yield S by reduced leaf-restriction onto I, i.e. for which $\mathrm{red}(T|I) = S$. Then the following formulas hold.*

(i) The generalized differential $\partial_S(T)$ is given by a sum $\sum_{S'} \mu_{S'}(T)S'$ over trees S' which are complements, see (2.2.15), of S in T.

(ii) Furthermore, if S is of the form $S = \vee^2(S^1.S^2)$, and T is of the form $T = \vee^2(T^1.T^2)$ (with non-trivial S_i, T_i) then

$$\mu_{\vee^2(S^1.S^2)}(\vee^2(T^1.T^2)) = \mu_{\vee^2(S^1.S^2)}(T^1) + \mu_{\vee^2(S^1.S^2)}(T^2) + \mu_{S^1}(T^1)\mu_{S^2}(T^2).$$

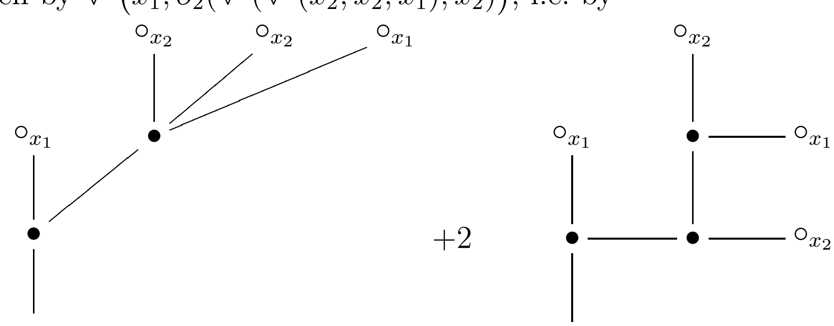
Proof. Formula (i) is a reformulation of Definition (4.3.6) in the case where $f = T$ is a monomial. Formula (ii) is an easy consequence of Proposition (4.3.7) (iii). □

Example 4.3.9. Let f be the monomial

$$\vee^2\Big(\vee^2\big(x_1, \vee^3(x_2, x_2, x_2)\big)\Big), \vee^2\big(\vee^3(x_2, x_2, x_1), x_2\big)\Big),$$

of Example (4.2.5).

Then $\partial_{\vee^4(x_2,x_2,x_2,x_2)}(f) = 0$. The generalized derivative $\partial_{\vee^2(\vee^3(x_2,x_2,x_2),x_2)}(f)$ is given by $\vee^2\big(x_1, \partial_2(\vee^2(\vee^3(x_2, x_2, x_1), x_2)\big)$, i.e. by



One notes that these terms also occur in $\frac{1}{4!}\big(\partial_2\big)^4(f)$.

More generally, for $x_k \in X$, if $Z := \{x_k\}$ and $W^{(n)} := \mathrm{PRTree}^n\{M^Z\}$ is the set of reduced trees with n leaves that are all labeled by x_k, then:

$$\sum_{S \in W^{(n)}} \partial_S = \frac{1}{n!}(\partial_k)^n.$$

This can be shown by a recursive argument, as

$$\frac{1}{n!}\big(\partial_k\big)^n(\vee^p(T^1 \dots T^p)) = \frac{1}{n!} \sum_{i_1+\ldots+i_p=n} \binom{n}{i_1, \dots, i_p} \vee^p \left(\big(\partial_k\big)^{i_1}(T^1) \dots \big(\partial_k\big)^{i_p}(T^p)\right),$$

$$\sum_{S \in W^{(n)}} \partial_S(\vee^p(T^1 \dots T^p)) = \sum_{i_1+\ldots+i_p=n} \quad \sum_{S^1 \in W^{(i_1)}, \dots, S^p \in W^{(i_p)}} \vee^p\big(\partial_{S^1}(T^1) \dots \partial_{S^p}(T^p)\big).$$

Lemma 4.3.10. *For f homogeneous of degree n, f is primitive if and only if $\partial_T(f) = 0$ for all monomials $T \in K\{X\}_\infty$ with $1 \le \deg T < \frac{n+1}{2}$.*

Proof. By Definition (4.3.6), f is primitive if and only if $\partial_T(f) = 0$ for all monomials $T \in K\{X\}_\infty$. Using the cocommutativity of Δ_a, the criterion follows. $\square$

4.4 Generalized Taylor expansions

Given any element f in a unitary $\mathcal{C}om$-algebra $K[x_1, \dots, x_m]/I$, the coefficients $a_{(n_1,\dots,n_m)} \in K$ in the formula

$$f = \sum a_{(n_1,\dots,n_m)} x_1^{n_1} \dots x_m^{n_m},$$

can be successively determined by a process of calculating highest non-vanishing partial derivatives $\partial_1^{n_1} \dots \partial_m^{n_m}$ (and doing subtractions).

A similar Taylor expansion also exists for noncommutative (cf. [Dre84], [Ger98]) and non-associative algebras (cf. [GH03], [DH03]).

We are going to describe it for the $\mathcal{M}ag_\omega$-algebra $K\{X\}_\infty$ or, more generally, for $\mathcal{M}ag_\omega$-algebras $A = K\{X\}_\infty/I$. Here X is the ordered set of variables $x_1 < x_2 < \dots$, and I is a multihomogeneous ideal which is invariant under all partial derivatives ∂_k. We denote the residue classes of $x_k \in X$ by the same symbols x_k, and we denote the induced partial derivatives on A also by ∂_k.

Lemma 4.4.1. *The subspace A_0 of A given by all $f \in A$ such that $\partial_k(f) = 0$ is a $\mathcal{M}ag_\omega$-algebra.*

Proof. The field K is contained in A_0. For $f_1, \dots, f_n \in A_0$, $\partial_k\big(\vee^n(f_1.f_2 \dots f_n)\big) = \sum_{i=1}^n \vee^n(\dots f_{i-1}.0.f_{i+1} \dots) = 0$, all k. Thus $\vee^n(f_1.f_2 \dots f_n) \in A_0$. $\square$

Definition 4.4.2. For $b \in A$, let $R_b : A \to A$, $a \mapsto a \cdot b := \vee^2(a, b)$ be the binary right multiplication by b.

For $f \in A$, $j \geq 0$, and $x_k \in X$, let

$$[f]_{\bullet} x_k^j = R_{x_k}^j(f) = (\dots((f \underbrace{\cdot x_k) \cdot x_k) \cdots x_k)}_{j}.$$

If $x_1, x_2, \dots, x_n \in X$, and $\mathbf{j} = (j_1, j_2, \dots, j_n) \in \mathbb{N}^n$, let

$$[f]_{\bullet} x^{\mathbf{j}} = \big(R_{x_n}^{j_n} \circ R_{x_{n-1}}^{j_{n-1}} \circ \dots \circ R_{x_1}^{j_1}\big)(f).$$

Proposition 4.4.3. *(i) There is a unique Taylor expansion of f with respect to x_k,*

$$f = \sum_{j \geq 0} [a_j^{(k)}(f)]_{\bullet} x_k^j,$$

for every $f \in A = K\{X\}_\infty / I$, with $\partial_k\big(a_j^{(k)}(f)\big) = 0$ (for all j).
The elements $a_j^{(k)}$ are homogeneous of degree $n - j$ if f is homogeneous of degree n.

(ii) There is a unique Taylor expansion

$$f = \sum_{\mathbf{j}} [a_{\mathbf{j}}(f)]_{\bullet} x^{\mathbf{j}}$$

for every $f \in A$, with $a_{\mathbf{j}}(f) \in A_0$.
The elements $a_{\mathbf{j}}(f)$ are homogeneous of degree $n - \sum_i j_i$ if f is homogeneous of degree n.
The map $\Phi : A \to A_0$ given by $f \mapsto a_{\mathbf{0}}(f)$ is a projector onto A_0.

(iii) If f depends only on the variables $x_1, x_2, \dots, x_m$, and $f = \sum_{\mathbf{j}} [a_{\mathbf{j}}]_{\bullet} x^{\mathbf{j}}$, then $f \cdot x_m$ is given by

$$\sum_{\mathbf{j} \in \mathbb{N}^{m-1} \times \mathbb{N}^*} [b_{\mathbf{j}}]_{\bullet} x^{\mathbf{j}}, \text{ where, for all } \mathbf{j},\ a_{\mathbf{j}} = b_{(j_0, \dots, j_{m-1}, 1 + j_m)}.$$

Especially $f \cdot x_m$ is in the kernel of Φ.

Proof. 1) For any $a_j^{(k)} \in A, j \geq 0$, with $\partial_k(a_j^{(k)}(f)) = 0$,

$$\partial_k \sum_{j \geq 0} [a_j^{(k)}]_{\bullet} x_k^j = \sum_{j \geq 0} [(j+1) a_{j+1}^{(k)}]_{\bullet} x_k^j,$$

by the Leibniz rule.
For $f \in A$, let $n = n(f)$ be the maximal $n \in \mathbb{N} \cup \{-\infty\}$ such that $\partial_k(f) \neq 0$ (or $-\infty$ for $f = 0$). Clearly $n \leq \deg f$.

We prove the uniqueness and existence of the Taylor expansion (i) by induction on n. In the case $n = 0$, the Taylor expansion of f with respect to x_k is given by $a_0^{(k)}(f) = f$. It is unique, because $\partial_k(a \cdot x_k) \neq 0$ for all $a \neq 0$.
For $n \geq 1$, let $\partial_k f$ be given by the unique Taylor expansion $f = \sum_{j\geq 0}[b_j^{(k)}]_\bullet x_k^j$, where $b_j^{(k)} = a_j^{(k)}(\partial_k f)$.
Let $\tilde{f}$ be given by $\tilde{f} = \sum_{j\geq 1}[a_j^{(k)}]_\bullet x_k^j$, where the elements $a_j^{(k)}$ are given by $(j+1)a_{j+1}^{(k)} = b_j^{(k)}$. Since $\partial_k(f - \tilde{f}) = 0$, $a_0^{(k)} := f - \tilde{f} \in A_0$.
Thus the desired Taylor expansion of f is $\sum_{j\geq 0}[a_j^{(k)}]_\bullet x_k^j$.
By construction, for homogeneous f, we get homogeneous elements $a_j^{(k)}(f)$ of the asserted degree.

2) Let f depend only on the variables $x_1, \ldots, x_m$, and let $f = \sum_{j\geq 0}[a_j^{(m)}(f)]_\bullet x_m^j$ be its Taylor expansion with respect to x_m.
Continuing with the variables $x_{m-1}, x_{m-2}, \ldots, x_1$, we get the desired Taylor expansion
$$f = \sum_{\mathbf{j}}[\ldots[a_{\mathbf{j}}(f)]_\bullet x_1^{j_1} \ldots]_\bullet x_m^{j_m} = \sum_{\mathbf{j}}[a_{\mathbf{j}}(f)]_\bullet x^{\mathbf{j}}.$$
For $f \in A$, let now $\mathbf{n} = \mathbf{n}(f) \in \mathbb{N}^m \cup \{-\infty\}$ be lexicographically maximal such that $\partial_1^{n_1} \cdots \partial_m^{n_m}(f) \neq 0$, or $-\infty$ for $f = 0$.
To show the uniqueness of the Taylor expansion (ii) we assume that a nontrivial presentation $0 = f = \sum_{\mathbf{j}\in J}[a_{\mathbf{j}}(f)]_\bullet x^{\mathbf{j}}$ with nonzero $a_{\mathbf{j}}(f) \in A_0$ is given, with $J \neq \emptyset$. On the one hand, $\mathbf{n}$ is the maximal element in J. On the other hand, $\mathbf{n} = -\infty$, which is a contradiction.
Again, the assertion about homogeneous elements follows by construction.
For $a \in A_0$, $a = a_{\mathbf{0}}(a)$ is the unique Taylor expansion, thus Φ is surjective with $\Phi \circ \Phi = \Phi$.

3) Assertion (iii) follows easily from the construction of the Taylor expansion (ii).
□

Remark 4.4.4. It should be noted that there are a lot of other possibilities to define Taylor expansions for $A = K\{X\}_\infty/I$. The expansion given above makes use of binary right multiplications, analogously one can use left multiplications.

If we define
$$[f]_\infty x^{\mathbf{j}} := \vee^{(1+j_1+\ldots+j_n)}(f, \underbrace{x_1, \ldots, x_1}_{j_1}, \ldots, \underbrace{x_n, \ldots, x_n}_{j_n}),$$
we get a unique Taylor expansion
$$f = \sum_{\mathbf{j}}[a_{\mathbf{j}}]_\infty x^{\mathbf{j}}$$
for every $f \in A$, with $a_{\mathbf{j}} = a_{\mathbf{j}}(f) \in A_0$.

The right Taylor expansion of (4.4.3) and its left-analogon have the advantage that they can be restricted to $\mathcal{M}ag$-algebras as well. While, for $f \in K\{X\} \subseteq K\{X\}_\infty$, the expression $f = \sum_{\mathbf{j}}[a_{\mathbf{j}}(f)]_\bullet x^{\mathbf{j}}$ is defined in $K\{X\}$, this is not the case for the non-binary expansion $f = \sum_{\mathbf{j}}[a_{\mathbf{j}}]_\infty x^{\mathbf{j}}$.

Corollary 4.4.5. *(compare [DH03])*

Let $X = \{x_1, x_2, \dots, x_m\}$. Let A be of the form $K\{X\}_\infty/I$ or $A = K\{X\}/I$, where I is a multihomogeneous ideal invariant under all partial derivatives.

(i) The Hilbert series of the multigraded algebra A and its subalgebra A_0 of constants are related by

$$\mathrm{Hilb}(A, t_1, t_2, \dots, t_m) = \prod_{j=1}^{m} \frac{1}{1-t_j} \mathrm{Hilb}(A_0, t_1, t_2, \dots, t_m).$$

(ii) Furthermore, if the ideal I is invariant with respect to the action of $GL_m(K)$, then

$$A^{(n)} = \bigoplus_{j=0}^{n} A_0^{(j)} \otimes \underbrace{\boxed{}\boxed{}\boxed{\cdots}\boxed{}\boxed{}}_{n-j}, \text{ for all } n,$$

as modules of the general linear group.

Proof. We consider the case where A is of the form $K\{X\}_\infty/I$, the binary case being completely analogous.

If $v_k, k \in J$ is a multihomogeneous basis of the vector space A_0, then the Taylor expansion of Proposition (4.4.3) shows that A has a basis

$$[v_k]_\bullet x^{\mathbf{j}} = \left(R_{x_n}^{j_n} \circ R_{x_{n-1}}^{j_{n-1}} \circ \cdots \circ R_{x_1}^{j_1}\right)(v_k),\ k \in J, \mathbf{j} = (j_1, j_2, \dots, j_m) \in \mathbb{N}^m.$$

The vector space homomorphism $A \to A_0 \otimes_K K[x_1, \dots, x_m]$ given by

$$[v_k]_\bullet x^{\mathbf{j}}$$

is an isomorphism of multigraded vector spaces.

Moreover it respects the action of the general linear group.

Since $K[x_1, \dots, x_m]^{(n)}$ is the (module of the) trivial representation with Young diagram

$$\underbrace{\boxed{}\boxed{}\boxed{\cdots}\boxed{}\boxed{}}_{n},$$

assertion (ii) follows.

For assertion (i), we note that the Hilbert series of $K[x_1, \dots, x_m]$ is

$$\mathrm{Hilb}(K[x_1, \dots, x_m], t_1, \dots, t_m) = \prod_{j=1}^{m} \frac{1}{1-t_j},$$

and that the Hilbert series of the tensor product is equal to the product of the Hilbert series of the factors (cf. [Ufn]).

$\square$

Remark 4.4.6. To obtain the Taylor expansions (i) and (ii) of Proposition (4.4.3), we are going to use the following algorithm:

Let $f \in A$ depend only on the variables $x_1, x_2, \ldots, x_m$.

Let n be maximal such that $(\partial_m)^n f \neq 0$. Then $a_n^{(m)}(f) = \frac{1}{n!}(\partial_m)^n f$.

For $\tilde{f} := f - [a_n^{(m)}(f)]_\bullet x_m^j$, it holds that $(\partial_m)^n \tilde{f} = 0$, and we can use $\tilde{f}$ to obtain the coefficients $a_n^{(m)}(f), j < n$.

Analogously we continue with the variables $x_{m-1}, x_{m-2}, \ldots, x_1$.

Remark 4.4.7. Since the free $\mathcal{M}ag$-algebra $K\{x_1\}$ in one variable x_1 has the Hilbert series

$$1 + \frac{1-\sqrt{1-4t_1}}{2} = \sum_{n=0}^{\infty} c_n t_1^n, \text{ with } c_0 := 1,$$

it follows from Corollary (4.4.5) that the subalgebra $K\{x_1\}_0$ has the Hilbert series

$$(1-t_1)\Big(1 + \frac{1-\sqrt{1-4t_1}}{2}\Big) = 1 + \sum_{n=1}^{\infty}(c_n - c_{n-1})t_1^n = 1 + t_1^3 + 3t_1^4 + 9t_1^5 + \ldots$$

(compare [DH03], [GH03]).

Hence the kernel of $\Phi : K\{x_1\} \to K\{x_1\}_0$ consists of all $f \cdot x_1, f \in K\{x_1\}$, and a vector space basis of $K\{x_1\}_0^{(n)}$, $n \geq 2$, is given by the set

$$\{\Phi(T) : \ T = T^1 \cdot T^2 \in W^{(n)}, T^1, T^2 \neq 1, \text{ and } T^2 \neq x_1\}.$$

4.5 Associators and the non-associative Jacobi relation

We are going to use Taylor expansion to obtain the first primitive elements in $K\{X\}_\infty$ and $K\{X\}$.

In degrees $n = 0, 1, 2$, the homogeneous components of degree n agree for the vector spaces $K\{X\}_\infty$, $K\{X\}$, and $K\langle X\rangle$. We describe the constants, which are also primitives, in the next example. Then, for $n \geq 3$, we treat the two cases $\mathcal{M}ag$ and $\mathcal{M}ag_\omega$ separately.

Example 4.5.1. Clearly all homogeneous elements of degree 0 are constants, all elements of degree 1 are not constants.

The Taylor expansion of $x_{i_1} \cdot x_{i_2}, i_1 \leq i_2$ is given by $[1]_\bullet x_{i_1} x_{i_2} := [1]_\bullet x^{\mathbf{j}}$, where

$$j_k = \begin{cases} 2 & k = i_1 = i_2 \\ 1 & : i_1 \neq i_2, k \in \{i_1, i_2\} \\ 0 & : \text{else.} \end{cases}$$

The commutators $[x_{i_2}, x_{i_1}] = x_{i_2} \cdot x_{i_1} - x_{i_1} \cdot x_{i_2}$, $i_1 < i_2$, are constants. Thus the Taylor expansion of $f = x_{i_2} \cdot x_{i_1}$, $i_1 < i_2$, is given by

$$[a_{\mathbf{0}}(f)]_\bullet x^{\mathbf{0}} + [1]_\bullet x_{i_1} x_{i_2}$$

with $a_{\mathbf{0}}(f) = a_{\mathbf{0}}(x_{i_2} \cdot x_{i_1}) = [x_{i_2}, x_{i_1}]$.

In other words, looking at the representation of Σ_2 given by multilinear homogeneous elements of degree 2, the constant part corresponds to the sign representation

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\,.$$

The non-constant part (in degree 2) corresponds to the trivial representation.

Remark 4.5.2. By Lemma (4.3.10), the primitive elements of order ≥ 2 form a subspace of the space $(K\{X\}_\infty)_0$ of constants.

Furthermore it follows that all homogeneous elements of degree ≤ 3 in $(K\{X\}_\infty)_0$ are primitive.

Example 4.5.3. Let $f = x_1 \cdot (x_1 \cdot x_1) \in K\{X\} \subset K\{X\}_\infty$. Then 3 is the maximal n with $(\partial_1)^n f \neq 0$. Thus we set $a_3(f) = a_3^{(1)}(f) = \frac{1}{3!}(\partial_1)^3 f = 1$.

For $\tilde{f} := f - [a_3(f)]_\bullet x_1^3 = f - (x_1 \cdot x_1) \cdot x_1$ we repeat the step and find that $\tilde{f} \in A_0$. Thus the Taylor expansion of f is given by

$$a_0(f) = x_1 \cdot (x_1 \cdot x_1) - (x_1 \cdot x_1) \cdot x_1, \quad a_3(f) = 1, \text{ and } a_j(f) = 0 \text{ else.}$$

Analogously the Taylor expansion of $f = \vee^3(x_1.x_1.x_1) \in K\{X\}_\infty$ is given by

$$a_0(f) = \vee^3(x_1.x_1.x_1) - (x_1 \cdot x_1) \cdot x_1, \quad a_3(f) = 1, \text{ and } a_j(f) = 0 \text{ else.}$$

Definition 4.5.4. For $f, g, h \in K\{X\}_\infty$, we denote by

$$(f, g, h)_b = (f \cdot g) \cdot h - f \cdot (g \cdot h)$$

the (binary) associator. The corresponding operation is denoted by $(x_1, x_2, x_3)_b$.

We define as a ternary associator in $K\{X\}_\infty$ the element

$$(f, g, h)_t = (f \cdot g) \cdot h - \vee^3(f, g, h).$$

Proposition 4.5.5. *The spaces of binary and ternary operations of* $\mathrm{Prim}\mathcal{M}ag_\omega$ *can be described as follows:*

The space $\mathrm{Prim}\mathcal{M}ag_\omega(2)$ *is given by* $K \cdot [x_1, x_2]$.

The space $\mathrm{Prim}\mathcal{M}ag_\omega(3)$ *has dimension 14 and is generated (as a* Σ_3*-module) by the operations*

$$(x_1, x_2, x_3)_t, \ (x_1, x_2, x_3)_b, \text{ and } [[x_1, x_2], x_3]$$

which fulfill $[[x_2, x_1], x_3] = -[[x_1, x_2], x_3]$ *and the equation*

$$\begin{aligned} [[x_1, x_2], x_3] + [[x_3, x_1], x_2] + [[x_2, x_3], x_1] = \ & (x_1, x_2, x_3)_b - (x_2, x_1, x_3)_b \\ & + (x_3, x_1, x_2)_b - (x_1, x_3, x_2)_b \\ & + (x_2, x_3, x_1)_b - (x_3, x_2, x_1)_b \end{aligned}$$

(called the non-associative Jacobi relation, compare [Lod03b], and also [SU02]).

Moreover, in terms of representations of the symmetric groups,

$$\mathrm{Prim}\mathcal{M}ag_\omega(3) = 2\ \ydiagram{3} \oplus 5\ \ydiagram{2,1} \oplus 2\ \ydiagram{1,1,1}\ .$$

Proof. 1) We use that all constant homogeneous elements of degree 3 are primitive, and apply Corollary (4.4.5). Let $A = K\{X\}_\infty$.
The subspace of $A^{(3)}$ orthogonal to the space $A_0^{(3)}$ of primitive elements is

$$\ydiagram{3} \oplus \left(\ydiagram{1,1} \otimes \ydiagram{1} \right),$$

where we use Young diagrams instead of GL-modules. The same description applies to the corresponding Σ_3-modules (which contain only multilinear elements).
By Young's rule,

$$\ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{2,1} \oplus \ydiagram{1,1,1}\ .$$

2) By comparison with three copies of the regular representation, one immediately arrives at the description of $\mathrm{Prim}\mathcal{M}ag_\omega(3)$ as a representation of Σ_3 given above. Counting dimensions yields $14 = 2 + 5 \cdot 2 + 2$.

3) The commutators $[x_1, x_2] = -[x_2, x_1]$ and associators $(x_1, x_2, x_3)_b$, $(x_1, x_2, x_3)_t$ are primitive.
If we define $a(x_1, x_2, x_3) := [[x_1, x_2], x_3] - (x_1, x_2, x_3)_b + (x_2, x_1, x_3)_b$, then

$$a(x_1, x_2, x_3) = x_3 \cdot (x_2 \cdot x_1) - x_3 \cdot (x_1 \cdot x_2) + x_1 \cdot (x_2 \cdot x_3) - x_2 \cdot (x_1 \cdot x_3).$$

The latter sum contains only right normed parenthesized words.
Thus the non-associative Jacobi relation

$$0 = a(x_1, x_2, x_3) + a(x_3, x_1, x_2) + a(x_2, x_3, x_1)$$

holds, simply because the classical Jacobi relation holds.

□

Proposition 4.5.6. *The spaces of binary and ternary operations of* $\mathrm{Prim}\mathcal{M}ag$ *can be described as follows:*

The space $\mathrm{Prim}\mathcal{M}ag(2)$ *is given by* $K \cdot [x_1, x_2]$.

The space $\mathrm{Prim}\mathcal{M}ag(3)$ *has dimension 8 and is generated by the operations*

$$(x_1, x_2, x_3)_b \text{ and } [[x_1, x_2], x_3]$$

which fulfill $[[x_2, x_1], x_3] = -[[x_1, x_2], x_3]$ *and the non-associative Jacobi relation.*

Moreover, in terms of representations of the symmetric groups,

$$\mathrm{Prim}\mathcal{M}ag(3) = \ydiagram{3} \oplus\ 3\ \ydiagram{2,1} \oplus \ydiagram{1,1,1}\ .$$

Proof. The proof is very similar to the proof of Proposition (4.5.5).

The only difference is the following: In step 2 of the proof, we have to compare with two instead of three copies of the regular representation to obtain $\mathrm{Prim}\mathcal{M}ag(3)$ as a representation of Σ_3. Counting dimensions yields $8 = 1 + 3 \cdot 2 + 1$.

□

Example 4.5.7. Let f be the tree monomial

$$(x_1 \cdot x_2) \cdot (x_3 \cdot x_4) = \vee^2(\vee^2(x_1.x_2). \vee^2 (x_1.x_2)) \in A = K\{X\} \subset K\{X\}_\infty.$$

Then the constant term of its Taylor expansion with respect to x_4 is the term

$$a_0^{(4)}(f) = (x_1 \cdot x_2) \cdot (x_3 \cdot x_4) - ((x_1 \cdot x_2) \cdot x_3) \cdot x_4,$$

which is a constant for the derivatives ∂_4 and ∂_3.

To get the constant term $a_{\mathbf{0}}(f) \in A_0$, we compute that

$$\partial_2\big(a_0^{(4)}(f)\big) = x_1 \cdot (x_3 \cdot x_4) - (x_1 \cdot x_3) \cdot x_4 = -(x_1, x_3, x_4)_b,$$

$$\begin{aligned} \text{and} \quad & \partial_1\big((x_1 \cdot x_2) \cdot (x_3 \cdot x_4) - ((x_1 \cdot x_2) \cdot x_3) \cdot x_4 + (x_1, x_3, x_4)_b \cdot x_2\big) \\ & = -(x_2, x_3, x_4)_b. \end{aligned}$$

Thus $a_{\mathbf{0}}(f)$ is given by

$$\begin{aligned} & p(x_1, x_2, x_3, x_4) := \\ & (x_1 \cdot x_2) \cdot (x_3 \cdot x_4) - ((x_1 \cdot x_2) \cdot x_3) \cdot x_4 + (x_1, x_3, x_4)_b \cdot x_2 + (x_2, x_3, x_4)_b \cdot x_1 \\ & = (x_1, x_3, x_4)_b \cdot x_2 + (x_2, x_3, x_4)_b \cdot x_1 - ((x_1 \cdot x_2), x_3, x_4)_b. \end{aligned}$$

Let S be one of the six two-leaf tree monomials $x_i \cdot x_j$ with $i, j \in \underline{3}$, $i \neq j$. Then it is immediately checked that the generalized differential operator ∂_S applied to $p(x_1, x_2, x_3, x_4)$ yields zero. (In fact, the corresponding expression in associative variables is zero, and ∂_S evaluated on degree 4 elements cannot distinguish between different bracketings). Therefore, by Lemma (4.3.10), $p(x_1, x_2, x_3, x_4)$ is primitive.

Analogously, using the Taylor expansion of left multiplications, one can show that the following element is primitive:

$$q(x_1,x_2,x_3,x_4) = \\ (x_1\cdot x_2)\cdot(x_3\cdot x_4) - x_1\cdot(x_2\cdot(x_3\cdot x_4)) - x_3\cdot(x_1,x_2,x_4)_b - x_4\cdot(x_1,x_2,x_3)_b.$$

By adding the primitive element $[x_4,(x_1,x_2,x_3)_b]$ to $q(x_1,x_2,x_3,x_4)$ we get the non-zero primitive element $(x_1,x_2,x_3\cdot x_4)_b - x_3\cdot(x_1,x_2,x_4)_b - (x_1,x_2,x_3)_b\cdot x_4$.

We also note that

$$\begin{aligned}(x_1,x_2\cdot x_3,x_4)_b &- x_2\cdot(x_1,x_3,x_4)_b - (x_1,x_2,x_4)_b\cdot x_3\\ &= q(x_1,x_2,x_3,x_4) + [(x_1,x_3,x_4)_b,x_2] + [(x_2,x_3,x_4)_b,x_1]\\ &\quad - p(x_1,x_2,x_3,x_4) - [(x_1,x_2,x_3)_b,x_4] - [(x_1,x_2,x_4)_b,x_3].\end{aligned}$$

Corollary 4.5.8. *The operad* Prim$\mathcal{M}ag$ *(and also the operad* Prim$\mathcal{M}ag_\omega$*) cannot be generated by quadratic and ternary operations.*

Proof. By Example (4.5.7), $p(x_1,x_2,x_3,x_4)$ is an element of Prim$\mathcal{M}ag(4)$.

It cannot be generated with respect to insertion by the binary and ternary operations given in Proposition (4.5.5), because $p(x_1,x_1,x_1,x_1)\neq 0$ and $(x_1\cdot x_2)\cdot(x_3\cdot x_4)$ occurs in the support of $p(x_1,x_2,x_3,x_4)$.

□

Remark 4.5.9. It was stated in [GH03] that the element q is primitive. In [SU02], Shestakov and Umirbaev recursively construct similar primitive elements

$$P(x^{\mathbf{i}},y^{\mathbf{j}},z) = (x^{\mathbf{i}},y^{\mathbf{j}},z)_b - {\sum}'(x^{\mathbf{i}})_{(1)}\cdot(y^{\mathbf{j}})_{(1)}\cdot P\big((x^{\mathbf{i}})_{(2)},(y^{\mathbf{j}})_{(2)},z\big),\quad x^{\mathbf{i}} := [1]_\bullet x^{\mathbf{i}}$$

They also show that this gives a complete set of primitive operations, and they pose the problem to give an intrinsic characterization of the set of primitive elements in terms of primitive operations.

4.6 The shuffle multiplication

We are going to describe the graded dual $K\{X\}_\infty^{*g}$ of the $\mathcal{M}ag_\omega$-Hopf algebra $K\{X\}_\infty$ equipped with co-addition.

In analogy to the classical case, see Remark (3.3.8), the commutative associative binary operation corresponding to Δ_a is called planar tree shuffle multiplication (or planar shuffle product, see [Ger04b]).

Proposition 4.6.1. *The vector spaces $K\{X\}_\infty$ and $K\{X\}_\infty^{*g}$ can be identified by mapping the basis given by tree monomials T on the corresponding dual basis elements δ_T. Then the commutative associative multiplication Δ_a^{*g} is given by the binary*

operation

$$\begin{aligned} \sqcup\!\sqcup : & K\{X\}_\infty \otimes K\{X\}_\infty \to K\{X\}_\infty \\ & T^1 \otimes T^2 \mapsto \sum_{\textit{all shuffles } T \textit{ of } T^1 \textit{ and } T^2} c_T\, T \quad \textit{for tree monomials } T^1, T^2, \end{aligned}$$

where $c_T \geq 1$ *is the number of subsets* $I \subseteq \mathrm{Le}(T)$ *with*

$$\mathrm{red}(T|I) = T^1, \mathrm{red}(T|I^c) = T^2.$$

Especially $1 \sqcup\!\sqcup 1 = 1$.

Proof. The space $K\{X\}_\infty \cong K\{X\}_\infty^{*g}$ together with the binary operation $\sqcup\!\sqcup$ given by Δ_a^{*g} is a $\mathcal{C}om$-algebra with unit 1, see Lemma (3.3.10) and Remark (3.3.11).

The multiplication $\sqcup\!\sqcup$ is characterized by the formula

$$\langle T^1 \sqcup\!\sqcup T^2, T\rangle = \langle T^1 \otimes T^2, \Delta_a(T)\rangle,$$

where $\langle S^1 \otimes S^2, T^1 \otimes T^2\rangle = \langle S^1, T^1\rangle \otimes \langle S^2, T^2\rangle$, and $\langle S, T\rangle = \begin{cases} 1 : S = T \\ 0 : S \neq T \end{cases}$, for T, T^1, T^2, S, S^1, S^2 tree monomials.

By Proposition (4.3.5),

$$\Delta_a(T) = \sum_{I \subseteq \mathrm{Le}(T)} \mathrm{red}(T|I) \otimes \mathrm{red}(T|I^c).$$

Therefore the number c_T of subsets $I \subseteq \mathrm{Le}(T)$ with

$$\mathrm{red}(T|I) = T^1, \mathrm{red}(T|I^c) = T^2$$

is equal to $\langle T^1 \sqcup\!\sqcup T^2, T\rangle = \langle T^1 \otimes T^2, \Delta_a(T)\rangle$, and $c_T \geq 1$ if and only if T is a shuffle of T^1 and T^2, see Definition (2.2.15).

□

Remark 4.6.2. Obviously, if T^1 has k leaves labeled bijectively by $x_1, \dots, x_k$, and T^2 has $n-k$ leaves labeled bijectively by $x_{k+1}, \dots, x_n$, then only $c_T = 1$ can occur, thus

$$T^1 \sqcup\!\sqcup T^2 = \sum_{\text{all shuffles } T \text{ of } T^1 \text{ and } T^2} T$$

and we get a generalization of the well-known shuffle multiplication of permutations $\sqcup\!\sqcup : K\Sigma_k \times K\Sigma_{n-k} \to K\Sigma_n$.

Example 4.6.3. For T in PRTree^n a (non-labeled) planar reduced tree with n leaves, let $T^{12\dots n}$ be the corresponding tree in $\mathrm{PRTree}\{M^X\}$ with first leaf labeled by x_1, second by x_2, and so on.

For $\sigma \in \Sigma_n$, let $\sigma T^{12\ldots n}$ be the tree which is obtained by permuting the labels of the leaves, i.e. with first leaf labeled by $x_{\sigma(1)}$, second by $x_{\sigma(2)}$, and so on.

Then the product $x_1 \sqcup\!\sqcup x_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup x_n$ is given by

$$\sum_{T \in \mathrm{PRTree}^n} \sum_{\sigma \in \Sigma_n} \sigma T^{12\ldots n}$$

as can be shown by induction (the case $n = 1$ being trivial) using the fact that every term of $x_1 \sqcup\!\sqcup x_2 \sqcup\!\sqcup \ldots \sqcup\!\sqcup x_n$ occurs in a unique way as a shuffle of a term of $x_1 \sqcup\!\sqcup x_2 \sqcup\!\sqcup \ldots x_{n-1}$ and x_n.

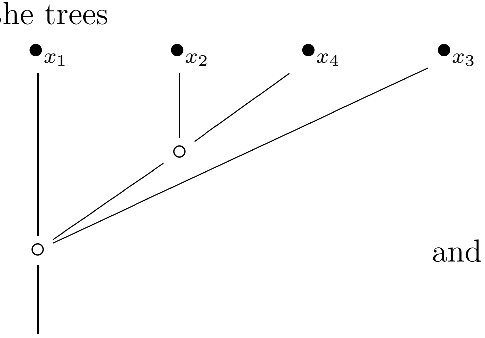
Example 4.6.4. Let $f = [x_2, x_1]$. Then, by a similar argument as in (4.6.3), we get

$$f \sqcup\!\sqcup x_3 = \sum_{T \in \mathrm{PRTree}^3} T^{321} + T^{231} + T^{213} - T^{123} - T^{132} - T^{312}.$$

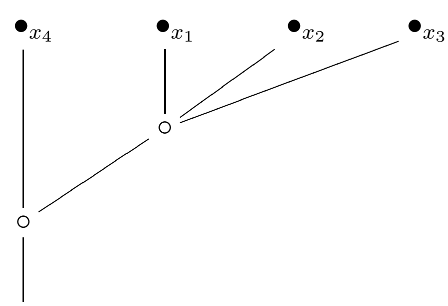
If $f = \vee^3(x_1.x_2.x_3)$, then $f \sqcup\!\sqcup x_4$ is given by

$$\begin{aligned}
&\vee^4(x_1.x_2.x_3.x_4) + \vee^4(x_1.x_2.x_4.x_3) + \vee^4(x_1.x_4.x_2.x_3) + \vee^4(x_4.x_1.x_2.x_3)\\
&+ \vee^3(\vee^2(x_1.x_4).x_2.x_3) + \vee^3(x_1.\vee^2(x_2.x_4).x_3) + \vee^3(x_1.x_2.\vee^2(x_3.x_4))\\
&+ \vee^3(\vee^2(x_4.x_1).x_2.x_3) + \vee^3(x_1.\vee^2(x_4.x_2).x_3) + \vee^3(x_1.x_2.\vee^2(x_4.x_3))\\
&+ \vee^2(\vee^3(x_1.x_2.x_3).x_4) + \vee^2(x_4.\vee^3(x_1.x_2.x_3)).
\end{aligned}$$

In this sum, the shuffles $\vee^3(x_1.\vee^2(x_2.x_4).x_3)$ and $\vee^2(x_4.\vee^3(x_1.x_2.x_3))$ correspond to the trees



and



Proposition 4.6.5. *The graded dual of the $\mathcal{M}ag$-Hopf algebra $K\{X\}$ equipped with co-addition is the quotient of $(K\{X\}_\infty, \sqcup\!\sqcup)$ with respect to the projection*

$$\pi : K\{X\}_\infty \to K\{X\}$$

$$\textit{given on monomials } T \textit{ by } T \mapsto \begin{cases} T & : T\,\textit{binary} \\ 0 & : \textit{else.} \end{cases}$$

Proof. Under the identification $K\{X\}_\infty^{*g} = K\{X\}_\infty$ of Proposition (4.6.1), the injection $\iota : K\{X\} \to K\{X\}_\infty$ corresponds to the projection π defined above.

The injection ι can be viewed as a morphism of $\mathcal{M}ag_\omega$-Hopf algebras.

It follows that $(K\{X\}_\infty^{*g}/\ker\pi, \sqcup\!\sqcup)$, where $\sqcup\!\sqcup$ becomes an operation on binary trees, is the graded dual of $(K\{X\}, \Delta_a)$.

$\square$

4.7 An analogon of Poincaré-Birkhoff-Witt

As a vector space (in fact as a coalgebra) the free $\mathcal{A}s$-algebra on V is isomorphic to the free $\mathcal{C}om$-algebra generated by all Lie polynomials, i.e. by the primitive elements. This is the Poincaré-Birkhoff-Witt theorem.

To describe the operads Prim$\mathcal{M}ag$ and Prim$\mathcal{M}ag_\omega$, we are going to use an analogon of this theorem.

First, we need to describe the primitive elements as irreducible elements with respect to the shuffle multiplication, and we also need a description of the operation $(\vee^2)^* : K\{X\}_\infty \to K\{X\}_\infty \otimes K\{X\}_\infty$.

Proposition 4.7.1. *Let $f \in K\{X\}_\infty^{(n)}$ be homogeneous of degree $n \geq 1$.*

(i) If f is primitive, then $\langle f, g_1 \sqcup\!\sqcup g_2\rangle = 0$ for all homogeneous $g_1, g_2 \in K\{X\}_\infty$ of degree ≥ 1.

(ii) If S is a tree monomial of degree $1 \leq k \leq n-1$, then $\partial_S(f) = 0$ if and only if $\langle f, S \sqcup\!\sqcup g\rangle = 0$ for all $g \in K\{X\}_\infty^{(n-k)}$.

(iii) If the homogeneous element f of degree n is orthogonal to all shuffle products $S \sqcup\!\sqcup T$, for S, T tree monomials of degrees $k, n-k$ with $1 \leq k < \frac{n+1}{2}$, then f is primitive.

The analogous assertions hold in the binary case $f \in K\{X\}^{(n)}$.

Proof. We consider elements of $K\{X\}_\infty$ (the proof for $K\{X\}$ is almost literally the same). By definition,

$$\langle \Delta_a(f), g_1 \otimes g_2\rangle = \langle f, g_1 \sqcup\!\sqcup g_2\rangle.$$

Clearly, if f is primitive, $\langle f, g_1 \sqcup\!\sqcup g_2\rangle = \langle f, g_1\rangle\langle 1, g_2\rangle + \langle 1, g_1\rangle\langle f, g_2\rangle = 0$.

If $\partial_S(f) \neq 0$ for some tree monomial S of degree $1 \leq k \leq n-1$, then by definition

$$0 \neq \langle \Delta_a(f), S \otimes g\rangle = \langle f, S \sqcup\!\sqcup g\rangle \text{ for some } g \in K\{X\}_\infty^{(n-k)}.$$

If f is orthogonal to all shuffle products $S \sqcup\!\sqcup S'$, S, S' tree monomials with $\deg S = k \geq 1$, then

$$\Delta_a(f) = \sum_{T \in \mathrm{PRTree}\{M^X\}_{-\{S\}}} T \otimes \partial_T(f)$$

and $\partial_S(f) = 0$.

Then assertion (iii) follows by Lemma (4.3.10).

□

Definition 4.7.2. Let $A = K\{X\}_\infty$ or $A = K\{X\}$.

We denote by $\nabla_k : A \to A^{\otimes k}, k \geq 2$ (for $A = K\{X\}$, $k = 2$ only) the maps given by $(\vee^k)^* : A^{*g} \to (A^{*g})^{\otimes k}$, see (3.3.10).

Lemma 4.7.3. *We consider the unitary $\mathcal{C}om$-algebras $A = (K\{X\}_\infty, \sqcup\!\sqcup)$ and $A = (K\{X\}, \sqcup\!\sqcup)$, and we denote by $\bar{A}$ the augmentation ideal.*

(i) The K-linear map $\nabla_2 : A \to A \otimes A$ is a morphism of $\mathcal{C}om$-algebras.

(ii) The K-linear map ∇_2 is non-coassociative. If T is an admissibly labeled tree, then

$$\nabla_2(T) = \begin{cases} T \otimes 1 + 1 \otimes T + T^1 \otimes T^2 & : T = \vee^2(T^1.T^2) \\ T \otimes 1 + 1 \otimes T & : ar_\rho \neq 2 \quad (\rho \text{ the root}). \end{cases}$$

(iii) If $\nabla_2'(f) := \nabla_2(f) - f \otimes 1 - 1 \otimes f$, then $\nabla_2'(f) \in \bar{A} \otimes \bar{A}$.

*In other words, these unitary $\mathcal{C}om$-algebras are co-**D** objects, where **D** is the category of unitary magmas.*

Proof. 1) The chosen unit actions on $\mathcal{M}ag_\omega$ (and also on $\mathcal{M}ag$) and $\mathcal{C}om$ have the property that the operations on the tensor product $A \otimes A$ are defined component-wise. We have to check that

$$\nabla_2 \circ \sqcup\!\sqcup = (\sqcup\!\sqcup \otimes \sqcup\!\sqcup) \circ \tau_2 \circ (\nabla_2 \otimes \nabla_2).$$

By looking at the graded dual, this is equivalent to the equation

$$\Delta \circ \vee^2 = (\vee^2 \otimes \vee^2) \circ \tau_2 \circ (\Delta \otimes \Delta).$$

The latter equation is fulfilled, because $(\vee^2 \otimes \vee^2) \circ \tau_2$ is $\vee^2_{A \otimes A}$ and Δ is a morphism of $\mathcal{M}ag$-algebras.

2) Since ∇_2 is determined by the equation

$$\langle \vee^2(f_1.f_2), g\rangle = \langle f_1 \otimes f_2, \nabla_2(g)\rangle$$

we conclude that, for T an admissibly labeled tree,

$$\nabla_2(T) = T \otimes 1 + 1 \otimes T + \sum_{T = \vee^2(T^1.T^2)} T^1 \otimes T^2.$$

It also follows that ∇_2 is not coassociative. We note that for $f, g \in A$,

$$\langle f, g\rangle = \langle \vee^2(1.f), g\rangle = \langle 1 \otimes f, \nabla_2(g)\rangle = \langle f \otimes 1, \nabla_2(g)\rangle$$

Thus assertion (iii) follows.

3) The categorical coproduct for $\mathcal{C}om$-algebras is the tensor product $\otimes$. By (i) and (iii), ∇_2 provides the unitary $\mathcal{C}om$-algebras $A = (K\{X\}_\infty, \sqcup\!\sqcup)$ and $A = (K\{X\}, \sqcup\!\sqcup)$ with the structure of a co-magma object, see Section (3.1), with counit given by the augmentation map.

$\square$

Remark 4.7.4. The vector space $K\{X\}_\infty$ (or $K\{X\}$) equipped with ∇_2 is a non-associative coalgebra in the sense of ([ACM94], [Gri88]).

Theorem 4.7.5. *Let either $A = K\{X\}_\infty$ or $A = K\{X\}$, and let $W = \mathrm{Prim}A$ be its space of primitive elements, graded by $W^{(k)} = \mathrm{Prim}A^{(k)}$.*

(i) The unitary $\mathcal{C}om$-algebra $(A, ⧢)$ is freely generated by $W = \mathrm{Prim}A$ with respect to the shuffle multiplication.

(ii) The generating series of $\mathrm{Prim}\mathcal{M}ag_\omega$ is the logarithm $\log(1+t)$ of the generating series of $\mathcal{M}ag_\omega$. Similarly, the generating series of $\mathrm{Prim}\mathcal{M}ag$ is the logarithm of the generating series of $\mathcal{M}ag$.

(iii) Let $n \geq 2$, and let $B^{(n)}$ be the orthogonal complement of $\mathrm{Prim}A^{(n)}$ in the vector space $A^{(n)}$. If $f_1, \ldots, f_r$ is a basis (consisting of homogeneous elements) of $\oplus_{k=1}^{n-1}\mathrm{Prim}A^{(k)}$, then the elements

$$f_{i_1}⧢f_{i_2}⧢\ldots⧢f_{i_k}, 1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq r, \text{ with } \sum_{j=1}^{k} \deg f_{i_j} = n$$

form a basis of $B^{(n)}$.

Proof. 1) We have shown in Proposition (4.7.1) that the homogeneous primitive elements of A are also the homogeneous ⧢-irreducible elements of A, i.e. the elements f not of the form $g_1⧢g_2$ (for f, g_1, g_2 homogeneous of degree ≥ 1). Thus the unitary $\mathcal{C}om$-algebra morphism $\pi : K1 \oplus F_{\mathcal{C}om}(W) \to (A, ⧢)$ is surjective, and for every proper subspace U of W, $K1 \oplus F_{\mathcal{C}om}(U)$ cannot be isomorphic to $(A, ⧢)$.

2) We have to show that π is injective, i.e. that $(A, ⧢)$ is a free $\mathcal{C}om$-algebra. By Lemma (4.7.3), there exists a morphism $\nabla_2 : A \to A \otimes A$ of $\mathcal{C}om$-algebras, which provides the unitary $\mathcal{C}om$-algebras $(A, ⧢)$ with the structure of a co-**D** object in the category of $\mathcal{C}om$-algebras, where **D** is the category of unitary magmas.
Over a field K of characteristic 0, all connected (i.e. $A^{(0)} = K$) $\mathcal{C}om$-algebras that are equipped with the structure of a unital co-magma are free. This is the Leray theorem, see [Oud99]. Thus π is an isomorphism.

3) We recall from Example (1.3.2) that the generating series of the operad $\mathcal{C}om$ is $\exp(t) - 1$. Combining this with assertion (i), we get that

$$f^{\mathcal{M}ag} = \exp\big(f^{\mathrm{Prim}\mathcal{M}ag}\big) - 1,$$
$$f^{\mathcal{M}ag_\omega} = \exp\big(f^{\mathrm{Prim}\mathcal{M}ag_\omega}\big) - 1,$$

Thus assertion (ii) follows.

4) Since the space Prim$A^{(n)}$ is orthogonal (with respect to $\langle , \rangle$) to the shuffle products in $A^{(n)}$, see Proposition (4.7.1), assertion (iii) follows from assertion (i).

$\square$

Theorem 4.7.6. *Let $A = K\{X\}_\infty$ or $A = K\{X\}$, and let $W = \mathrm{Prim}A$ be its space of primitive elements.*

Then (A, Δ_a) is a cofree (co-)nilpotent $\mathcal{C}om$-coalgebra, in the sense of (1.3.10) and (1.3.12), (co-)generated by W.

Proof. We introduced $(A, \sqcup\!\sqcup)$ as the graded dual of (A, Δ_a). Thus it is allowed to dualize, and the statement of the theorem is exactly dual to the one of Theorem (4.7.5)(i).

$\square$

Remark 4.7.7. Recall that Prim$\mathcal{A}s = \mathcal{L}ie$ and Prim$\mathcal{C}om$ = Vect with generating series $f^{\mathcal{L}ie}(t) = \sum_{n=1}^{\infty} \frac{(n-1)!}{n!} t^n = -\log(1-t)$ and $f^{\mathrm{Vect}}(t) = t$.

It is easy to check that $f^{\mathcal{A}s}(t) = \frac{1}{1-t} - 1 = \exp(f^{\mathcal{L}ie}(t)) - 1$ and $f^{\mathcal{C}om}(t) = \exp(f^{\mathrm{Vect}}(t)) - 1$, which are the analogues of Theorem (4.7.5)(ii) for the operads $\mathcal{A}s$ and $\mathcal{C}om$.

Theorem (4.7.5)(iii) shows that one can define orthogonal projectors

$$e_n^{(1)}, e_n^{(2)}, \ldots, e_n^{(n)}$$

which are similar to the Eulerian idempotents (see [Lod94]). The idempotent $e_n^{(i)}$ projects elements of $A^{(n)}$ to the subspace of i-factor shuffle products of primitive elements.

To obtain an explicit form of Eulerian idempotents for $\mathcal{M}ag$ and $\mathcal{M}ag_\omega$, one may use the non-associative (planar) exponential series of [Ger04a] (which maps primitive elements to group-like elements).

Remark 4.7.8. We are going to give explicit formulas for the generating series of Prim$\mathcal{M}ag$ and Prim$\mathcal{M}ag_\omega$ in the next section.

If $\mathcal{M}ag$ and $\mathcal{M}ag_\omega$ are replaced by $\mathcal{C}mg$ and the commutative version $\mathcal{C}mg_\omega$ of $\mathcal{M}ag_\omega$, Theorem (4.7.5) remains true, with the exception that we do not get explicit formulas for the generating series of the operads of primitives, see Remark (2.1.4).

For $\mathcal{P}$ = Prim$\mathcal{C}mg$, we get that $\mathcal{P}(2) = 0$ and $\mathcal{P}(3)$ is 2-dimensional, generated by the operation $(x_1, x_2, x_3)_b = (x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3) = -(x_3, x_2, x_1)_b$ subject to the relation $(x_1, x_2, x_3)_b + (x_2, x_3, x_1)_b + (x_3, x_1, x_2)_b = 0$.

Since the coefficient of t^4 in $\exp(t + \frac{2t^3}{3!})$ is $\frac{1}{4!}(1+8)$, we get that $\dim \mathcal{P}(4) = \dim \mathcal{C}mg(4) - 9 = 15 - 9 = 6$.

4.8 The generating series and representations

In the following, we describe the series

$$f^{\mathrm{Prim}\mathcal{M}ag} \text{ and } f^{\mathrm{Prim}\mathcal{M}ag_\omega}$$

explicitly. Moreover, we compute $\mathrm{Prim}\mathcal{M}ag_\omega(4)$ and $\mathrm{Prim}\mathcal{M}ag(4)$ as representations of Σ_4.

We start with the operad $\mathrm{Prim}\mathcal{M}ag$ and obtain from Theorem (4.7.5) the following corollary:

Corollary 4.8.1. *The dimension of* $\mathrm{Prim}\mathcal{M}ag(n)$ *is given by*

$$\dim \mathrm{Prim}\mathcal{M}ag(n) = (n-1)!c'_n$$

where c'_n *is the* n*-th* log*-Catalan number, see Example (2.1.9).*

Thus the sequence $\mathrm{Prim}\mathcal{M}ag(n), n \geq 1$, *starts with*

$$\begin{array}{cccccccccc} & 0!{\cdot}1, & 1!\cdot 1, & 2!\cdot 4, & 3!\cdot 13, & 4!\cdot 46, & 5!\cdot 166, & 6!\cdot 610, & \dots \\ \textit{i.e.} & 1, & 1, & 8, & 78, & 1104, & 19920, & 439200, & \dots \end{array}$$

Proof. By Theorem (4.7.5)(ii), we get that the generating series $f^{\mathrm{Prim}\mathcal{M}ag}(t)$ is given by

$$\log\Big(\frac{3-\sqrt{1-4t}}{2}\Big).$$

Thus, for $\sum_{n=1}^\infty \frac{b_n}{n!}\, t^n := f^{\mathrm{Prim}\mathcal{M}ag}(t)$,

$$\begin{aligned} \sum_{n=1}^\infty \frac{b_n}{(n-1)!} t^n &= t\cdot \frac{\partial}{\partial t} f^{\mathrm{Prim}\mathcal{M}ag}(t) \\ &= \frac{2t}{3\sqrt{1-4t}-1+4t} \;=\; \sum_{n=1}^\infty c'_n t^n, \end{aligned}$$

where c'_n is the n-th log-Catalan number of Example (2.1.9). □

Remark 4.8.2. We have already seen in Proposition (4.5.6), that the representation of Σ_n given by $\mathrm{Prim}\mathcal{M}ag(n)$ is not given by copies of the $(n-1)!$-dimensional representation $\mathcal{L}ie(n)$. For $n = 4$, we obtain the following description:

Proposition 4.8.3. *In terms of representations of the symmetric groups,*

$$\mathrm{Prim}\mathcal{M}ag(4) =$$

$$3\ \ydiagram{4} \;\oplus\; 10\ \ydiagram{3,1} \;\oplus\; 6\ \ydiagram{2,2} \;\oplus\; 10\ \ydiagram{2,1,1} \;\oplus\; 3\ \ydiagram{1,1,1,1}\,.$$

Proof. Let $A = K\{X\}$. We determine the space $A_0^{(4)}$ of degree 4 homogeneous constant elements (as a representation of the general linear or symmetric group) first.

We recall that $A_0^{(0)} = K$, $A_0^{(1)} = 0$, and

$$A_0^{(2)} \otimes \ydiagram{2} = \ydiagram{1,1} \otimes \ydiagram{2} = \ydiagram{3,1} \oplus \ydiagram{2,1,1} .$$

Furthermore, by Proposition (4.5.6),

$$A_0^{(3)} \otimes \ydiagram{1} \text{ is equal to}$$

$$\ydiagram{4} \oplus 4\,\ydiagram{3,1} \oplus 3\,\ydiagram{2,2} \oplus 4\,\ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1} .$$

Next we have to compare

$$\bigoplus_{j=0}^{3} A_0^{(j)} \otimes \underbrace{\boxed{}\boxed{\cdots}\boxed{}}_{4-j}$$

with $c_4 = 5$ copies of the regular representation

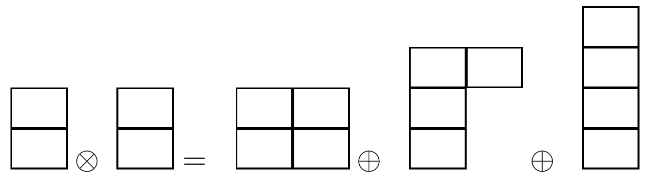
$$\ydiagram{4} \oplus 3\,\ydiagram{3,1} \oplus 2\,\ydiagram{2,2} \oplus 3\,\ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1} .$$

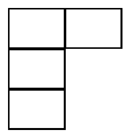
It follows that $A_0^{(4)}$ is given by

$$3\,\ydiagram{4} \oplus 10\,\ydiagram{3,1} \oplus 7\,\ydiagram{2,2} \oplus 10\,\ydiagram{2,1,1} \oplus 4\,\ydiagram{1,1,1,1} .$$

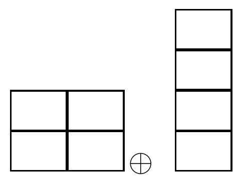
In view of Theorem (4.7.5), Prim$A$ is exactly the part of $A_0^{(4)}$ which is orthogonal to the shuffle products of two commutators $[x_i, x_j]$.

This is just the classical associative setting: In the representation of Σ_4 given by



the Σ_4-module  with vector space basis

$$[[x_1, x_2], [x_3, x_4]],\ \ [[x_1, x_3], [x_2, x_4]],\ \ [[x_1, x_4], [x_2, x_3]]$$

corresponds to the primitive part, and  corresponds to the shuffles.

Canceling out the part given by shuffle products, we arrive at the asserted sum of irreducible representations given by Prim$\mathcal{M}ag(4)$.

$\square$

Proposition 4.8.4. *Let $\sum_{\mathbf{i}} a_{\mathbf{i}}\ x_{i_1}.x_{i_2}\dots x_{i_n}, a_{\mathbf{i}} \in K$, be a homogeneous primitive element of degree n in $K\langle x_1, x_2, \dots, x_n\rangle$. Then*

$$\sum_{\mathbf{i}} a_{\mathbf{i}}\ (\dots((x_{i_1} \cdot x_{i_2}) \cdot x_{i_3}) \cdot \dots) \cdot x_{i_n} \ \textit{and} \ \sum_{\mathbf{i}} a_{\mathbf{i}}\ x_{i_1} \cdot (x_{i_2} \cdot (\dots \cdot (x_{i_{n-1}} \cdot x_{i_n}) \dots))$$

are homogeneous primitive elements of degree n in $K\{x_1, x_2, \dots, x_n\}$.

Especially, for $n \geq 3$, there are always (at least) two copies of the representation $\mathcal{L}ie(n)$ present in Prim$\mathcal{M}ag(n)$.

Proof. When we apply Δ_a to the terms of $f = \sum_{\mathbf{i}} a_{\mathbf{i}}\ (\dots((x_{i_1} \cdot x_{i_2}) \cdot x_{i_3}) \cdot \dots) \cdot x_{i_n}$, which are left-normed bracketed words, it is immediately seen that the result is given by tensor products of left-normed bracketed words. Therefore we do not need to pay attention to brackets when we want to check if f is primitive.

Completely similar is the case of right-normed bracketed words.

The actions of the general linear and the symmetric group do also keep left-normed (right-normed, respectively) brackets in order. Thus the assertion follows.

$\square$

Remark 4.8.5. With a similar argument, one can show that Prim$\mathcal{C}mg$ can be imbedded into Prim$\mathcal{M}ag$. More exactly, we can map primitive elements of the free $\mathcal{C}mg$-algebra $F_{\mathcal{C}mg}$ into the commutative subalgebra of $F_{\mathcal{M}ag}$ generated by the variables with respect to the multiplication $a \circ b := \frac{1}{2}(a \cdot b + b \cdot a)$.

Remark 4.8.6. We consider $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m), \lambda_1 \geq \dots \geq \lambda_m$, a partition of n (and the corresponding Young diagram with λ_1 boxes in the first row, ..., λ_m boxes in the last row).

To obtain a basis of the irreducible GL_n-modules (or Σ_n-modules) which are contained in $K\{X\}^{(n)}$ or in Prim$K\{X\}^{(n)}$, one can consider highest weight vectors (compare [Wey], [Dre], [Dre98] §1.2):

Definition 4.8.7. A polynomial $f \in K\{X\}^{(n)}$, of multi-degree $(\lambda_1, \lambda_2, \dots, \lambda_m)$ is said to be of highest weight, if

$$\partial_{ij}\big(f\big) = 0 \text{ all } 1 \leq j < i \leq m.$$

Remark 4.8.8. Each element of highest weight generates an irreducible GL_m-module in $K\{X\}^{(n)}$, and linearly independent elements of highest weight generate different copies.

Given a highest weight element $f(x_1, x_2, \dots, x_m)$ of multi-degree λ, the complete linearization, i.e. the multilinear component of

$$f(\sum_{i=1}^{\lambda_1} x_i, \sum_{i=\lambda_1+1}^{\lambda_1+\lambda_2} x_i, \dots, \sum_{i=n-\lambda_m}^{n} x_i),$$

is used to generate the corresponding Σ_n-module.

We consider the representation given by Prim$\mathcal{M}ag(4)$, see Proposition (4.8.3), and we may look for a basis of primitive highest weight elements.

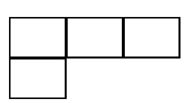
Example 4.8.9. For the five copies of the trivial representation occurring in the GL-module $K\{X\}^{(n)}$ we can take all five monomials of degree 5 in $K\{x_1\}^{(5)} \subset K\{X\}^{(5)}$ as linearly independent highest weight vectors.

There are only three linearly independent primitive highest weight vectors, namely

$$\Phi\big((x_1 \cdot x_1) \cdot (x_1 \cdot x_1)\big) = p(x_1, x_1, x_1, x_1)$$
$$\Phi\big(x_1 \cdot ((x_1 \cdot x_1) \cdot x_1)\big) = x_1 \cdot ((x_1 \cdot x_1) \cdot x_1) - 3(x_1 \cdot (x_1 \cdot x_1)) \cdot x_1 + 2((x_1 \cdot x_1) \cdot x_1) \cdot x_1$$
$$\Phi\big(x_1 \cdot (x_1 \cdot (x_1 \cdot x_1))\big) = x_1 \cdot (x_1 \cdot (x_1 \cdot x_1)) - 4(x_1 \cdot (x_1 \cdot x_1)) \cdot x_1 + 3((x_1 \cdot x_1) \cdot x_1) \cdot x_1$$

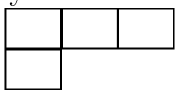
(see Remark (4.4.7)).

To compute the primitive highest weight vectors associated to the Young diagram



we have to compute with a basis consisting of 20 monomials, namely words $x_2x_1x_1x_1$, $x_1x_2x_1x_1$, $x_1x_1x_2x_1$, $x_2x_1x_1x_1$ with all 5 possible bracketings.

Since the operator ∂_{21} replaces x_2 by x_1 in these parenthesized words (and does not change the parentheses), the coefficient of $x_2x_1x_1x_1$ is determined by the coefficients of $x_2x_1x_1x_1$, $x_1x_2x_1x_1$, $x_1x_1x_2x_1$ (for every type of bracketing).

Under this condition, the requirement to be a constant (i.e. a primitive, in this case) yields five further equations for the coefficients. A basis for the space of solutions is given by the following ten primitive highest weight vectors associated to the Young diagram  :

$$\begin{aligned} f_1 =& x_2 \cdot (x_1 \cdot (x_1 \cdot x_1)) - 3x_1 \cdot (x_2 \cdot (x_1 \cdot x_1)) + 3x_1 \cdot (x_1 \cdot (x_2 \cdot x_1)) \\ & - x_1 \cdot (x_1 \cdot (x_1 \cdot x_2)) \end{aligned}$$

$$\begin{aligned} f_2 =& (x_1 \cdot x_1) \cdot (x_2 \cdot x_1) - x_1 \cdot (x_1 \cdot (x_2 \cdot x_1)) + x_1 \cdot (x_1 \cdot (x_1 \cdot x_2)) \\ & - (x_1 \cdot x_1) \cdot (x_1 \cdot x_2) \end{aligned}$$

$$\begin{aligned} f_3 =& x_2 \cdot (x_1 \cdot (x_1 \cdot x_1)) - x_2 \cdot ((x_1 \cdot x_1) \cdot x_1) + x_1 \cdot (x_2 \cdot (x_1 \cdot x_1)) \\ & - (x_1 \cdot x_2) \cdot (x_1 \cdot x_1) + 2x_1 \cdot ((x_1 \cdot x_2) \cdot x_1) - 2x_1 \cdot (x_1 \cdot (x_2 \cdot x_1)) \\ & - x_1 \cdot ((x_1 \cdot x_1) \cdot x_2) + (x_1 \cdot x_1) \cdot (x_1 \cdot x_2) \end{aligned}$$

$$\begin{aligned} f_4 =& x_2 \cdot (x_1 \cdot (x_1 \cdot x_1)) - x_2 \cdot ((x_1 \cdot x_1) \cdot x_1) - ((x_1 \cdot x_1) \cdot x_2) \cdot x_1 \\ & + (x_1 \cdot (x_1 \cdot x_2)) \cdot x_1 - x_1 \cdot (x_1 \cdot (x_1 \cdot x_2)) + x_1 \cdot ((x_1 \cdot x_1) \cdot x_2)) \\ & - (x_1 \cdot (x_1 \cdot x_1)) \cdot x_2 + ((x_1 \cdot x_1) \cdot x_1) \cdot x_2 \end{aligned}$$

$$\begin{aligned} f_5 =& 4x_2 \cdot (x_1 \cdot (x_1 \cdot x_1)) - 3x_2 \cdot ((x_1 \cdot x_1) \cdot x_1) - 3(x_1 \cdot x_2) \cdot (x_1 \cdot x_1) \\ & + 3((x_1 \cdot x_1) \cdot x_2) \cdot x_1 - 4x_1 \cdot (x_1 \cdot (x_1 \cdot x_2)) + 3x_1 \cdot ((x_1 \cdot x_1) \cdot x_2) \\ & + 3(x_1 \cdot x_1) \cdot (x_1 \cdot x_2) - 3((x_1 \cdot x_1) \cdot x_1) \cdot x_2 \end{aligned}$$

$$\begin{aligned} f_6 =& 4x_2 \cdot (x_1 \cdot (x_1 \cdot x_1)) - 3x_2 \cdot ((x_1 \cdot x_1) \cdot x_1) - (x_2 \cdot x_1) \cdot (x_1 \cdot x_1) \\ & - 2(x_1 \cdot x_2) \cdot (x_1 \cdot x_1) + 2((x_1 \cdot x_2) \cdot x_1) \cdot x_1 - 4x_1 \cdot (x_1 \cdot (x_1 \cdot x_2)) \\ & + 3x_1 \cdot ((x_1 \cdot x_1) \cdot x_2)) + 3(x_1 \cdot x_1) \cdot (x_1 \cdot x_2) - 2((x_1 \cdot x_1) \cdot x_1) \cdot x_2 \end{aligned}$$

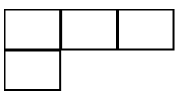
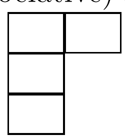
$$\begin{aligned} f_7 =& x_2 \cdot (x_1 \cdot (x_1 \cdot x_1)) - x_2 \cdot ((x_1 \cdot x_1) \cdot x_1) - x_1 \cdot (x_2 \cdot (x_1 \cdot x_1)) \\ & + (x_1 \cdot x_2) \cdot (x_1 \cdot x_1) + 2(x_1 \cdot (x_2 \cdot x_1)) \cdot x_1 - 2((x_1 \cdot x_2) \cdot x_1) \cdot x_1 \\ & + x_1 \cdot ((x_1 \cdot x_1) \cdot x_2) - (x_1 \cdot x_1) \cdot (x_1 \cdot x_2) - 2(x_1 \cdot (x_1 \cdot x_1)) \cdot x_2 \\ & + 2((x_1 \cdot x_1) \cdot x_1) \cdot x_2 \end{aligned}$$

$$\begin{aligned} f_8 =& 2x_2 \cdot (x_1 \cdot (x_1 \cdot x_1)) - x_2 \cdot ((x_1 \cdot x_1) \cdot x_1) - (x_2 \cdot x_1) \cdot (x_1 \cdot x_1) \\ & - 2x_1 \cdot (x_2 \cdot (x_1 \cdot x_1)) + 2x_1 \cdot ((x_2 \cdot x_1) \cdot x_1) - x_1 \cdot ((x_1 \cdot x_1) \cdot x_2) \\ & + (x_1 \cdot x_1) \cdot (x_1 \cdot x_2) \end{aligned}$$

$$\begin{aligned} f_9 =& 4x_2 \cdot (x_1 \cdot (x_1 \cdot x_1)) - 3x_2 \cdot ((x_1 \cdot x_1) \cdot x_1) - 3(x_2 \cdot x_1) \cdot (x_1 \cdot x_1) \\ &+ 2((x_2 \cdot x_1) \cdot x_1) \cdot x_1 - 4x_1 \cdot (x_1 \cdot (x_1 \cdot x_2)) + 3x_1 \cdot ((x_1 \cdot x_1) \cdot x_2) \\ &+ 3(x_1 \cdot x_1) \cdot (x_1 \cdot x_2) - 2((x_1 \cdot x_1) \cdot x_1) \cdot x_2 \end{aligned}$$

$$\begin{aligned} f_{10} =& 2x_2 \cdot (x_1 \cdot (x_1 \cdot x_1)) - 2x_2 \cdot ((x_1 \cdot x_1) \cdot x_1) - (x_2 \cdot x_1) \cdot (x_1 \cdot x_1) \\ &+ (x_2 \cdot (x_1 \cdot x_1)) \cdot x_1 - 2x_1 \cdot (x_1 \cdot (x_1 \cdot x_2)) + 2x_1 \cdot ((x_1 \cdot x_1) \cdot x_2) \\ &+ (x_1 \cdot x_1) \cdot (x_1 \cdot x_2) - (x_1 \cdot (x_1 \cdot x_1)) \cdot x_2. \end{aligned}$$

In the classical (associative) case, there is only one copy of this Young diagram present

in  $\oplus$  $= \mathcal{L}ie(4) = \mathrm{Prim}\mathcal{A}s(4)$, with highest weight vector

$$\begin{aligned} &x_2.x_1.x_1.x_1 - 3x_1.x_2.x_1.x_1 + 3x_1.x_1.x_2.x_1 - x_1.x_1.x_1.x_2 \\ &= [[[x_2, x_1], x_1], x_1]. \end{aligned}$$

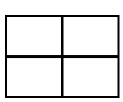
Under the projection onto the free $\mathcal{C}mg$-algebra, the ten elements f_i yield scalar multiples of only one highest weight vector (in commuting variables) given by

$$x_2 \cdot (x_1 \cdot (x_1 \cdot x_1)) + 2x_1 \cdot (x_1 \cdot (x_1 \cdot x_2)) - 3x_1 \cdot (x_2 \cdot (x_1 \cdot x_1)),$$

which is $p(x_2, x_1, x_1, x_1) - p(x_1, x_1, x_1, x_2)$. The image of $\Phi\big((x_1 \cdot x_1) \cdot (x_1 \cdot x_1)\big) = p(x_1, x_1, x_1, x_1)$ yields a nontrivial highest weight vector in commuting variables, too.

Similarly it can be checked that the image of $p(x_2, x_2, x_1, x_1) - p(x_1, x_2, x_2, x_1)$, see Example (4.5.7), is a highest weight vector (in commuting variables)

$$\begin{aligned} &(x_1 \cdot x_1) \cdot (x_2 \cdot x_2) - 2x_1 \cdot (x_1 \cdot (x_2 \cdot x_2)) - 2x_2 \cdot (x_2 \cdot (x_1 \cdot x_1)) \\ &- (x_1 \cdot x_2) \cdot (x_1 \cdot x_2) + 2x_1 \cdot (x_2 \cdot (x_1 \cdot x_2)) + 2x_2 \cdot (x_1 \cdot (x_1 \cdot x_2)) \end{aligned}$$

for  .

The complete linearizations of these three elements generate the Σ_4-module $\mathrm{Prim}\mathcal{C}mg(4)$, compare Remark (4.7.8).

Remark 4.8.10. A description of $\mathrm{Prim}\mathcal{M}ag$ by generators and relations would be interesting. We may take the generators of Proposition (4.5.6) together with the operations $p(x_1, x_2, x_3, x_4)$ and $q(x_1, x_2, x_3, x_4)$ given in Example (4.5.7) as the first generators.

Since

$$\begin{aligned} ([x_1, x_2], x_3, x_4)_b &= p(x_2, x_1, x_3, x_4) - p(x_1, x_2, x_3, x_4), \\ (x_1, x_2, [x_3, x_4])_b &= q(x_1, x_2, x_3, x_4) - q(x_1, x_2, x_4, x_3), \end{aligned}$$

we may replace $p(x_1, x_2, x_3, x_4)$ and $q(x_1, x_2, x_3, x_4)$ by

$$p'(x_1, x_2, x_3, x_4) := p(x_1, x_2, x_3, x_4) + p(x_2, x_1, x_3, x_4) \quad \text{and}$$
$$q'(x_1, x_2, x_3, x_4) := q(x_1, x_2, x_3, x_4) + q(x_1, x_2, x_4, x_3).$$

We get relations, e.g. when expressing the 12 elements $(x_{i_1}, [x_{i_2}, x_{i_3}], x_{i_4})_b, i_2 < i_3$, by $p'(x_1, x_2, x_3, x_4)$, $q'(x_1, x_2, x_3, x_4)$, and operations $[(x_1, x_2, x_3)_b, x_4]$.

We finish with the analogues of Corollary (4.8.1) and Proposition (4.8.3) for the case of $\mathrm{Prim}\mathcal{M}ag_\omega$.

Corollary 4.8.11. *The generating series* $f^{\mathrm{Prim}\mathcal{M}ag_\omega}(t)$ *is given by*

$$\log\Big(\frac{5 + t - \sqrt{1 - 6t + t^2}}{4}\Big).$$

Thus

$$\dim \mathrm{Prim}\mathcal{M}ag_\omega(n) = (n-1)!C'_n$$

where the integer sequence C'_n *is obtained from the super-Catalan numbers by logarithmic derivation. The sequence* $\dim \mathrm{Prim}\mathcal{M}ag_\omega(n), n \geq 1$, *starts with*

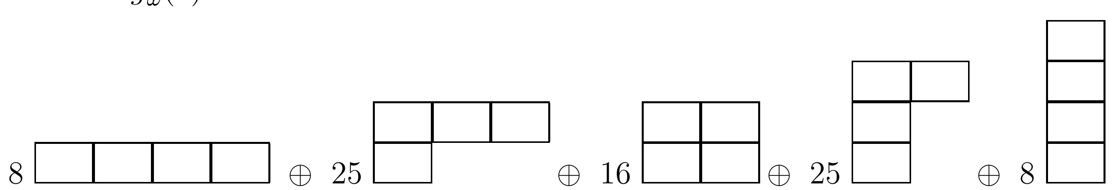
$$\begin{array}{llllllll} & 0!{\cdot}1, & 1!\cdot 1, & 2!\cdot 7, & 3!\cdot 33, & 4!\cdot 171, & 5!\cdot 901, & 6!\cdot 4831, \quad \ldots \\ \textit{i.e.} & 1, & 1, & 14, & 198, & 4104, & 108120, & 3478320, \quad \ldots \end{array}$$

Proof. The series $f^{\mathcal{M}ag_\omega}(t)$ is the generating series for the super-Catalan numbers. Analogously to the proof of Corollary (4.8.1) the assertion follows from Theorem (4.7.5)(ii).

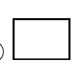
$\square$

Proposition 4.8.12. *In terms of representations of the symmetric groups,*

$\mathrm{Prim}\mathcal{M}ag_\omega(4) =$



Proof. The proof is similar to the proof of Proposition (4.8.3). For $A = K\{X\}_\infty$,

$A_0^{(3)} \otimes$  is equal to

$$2\,\ydiagram{4} \oplus 7\,\ydiagram{3,1} \oplus 5\,\ydiagram{2,2} \oplus 7\,\ydiagram{2,1,1} \oplus 2\,\ydiagram{1,1,1,1}\,.$$

We have to compare

$$\bigoplus_{j=0}^{3} A_0^{(j)} \otimes \underbrace{\boxed{}\boxed{\cdots}\boxed{}}_{4-j}$$

with $C_4 = 11$ copies of the regular representation.

It follows that $A_0^{(4)}$ is given by

$$8\,\ydiagram{4} \oplus 25\,\ydiagram{3,1} \oplus 17\,\ydiagram{2,2} \oplus 25\,\ydiagram{2,1,1} \oplus 9\,\ydiagram{1,1,1,1}\,,$$

for $A = K\{X\}_\infty$.

Next we cancel the shuffle products (of commutators) given in the proof of Proposition (4.8.3), and the assertion follows.

$\square$

4.9 Lazard-Lie theory for $\mathcal{P}$-Hopf algebras

There is a Lazard-Lie theory for complete cogroups over operads, see Ginzburg and Kapranov ([GK94]) and Fresse ([Fre98a]). Given a binary quadratic operad $\mathcal{P}$, complete cogroups over $\mathcal{P}$ are classified by algebras over the Koszul-dual operad $\mathcal{P}^!$. Originally, Lazard [Laz55] studied m-dimensional formal group laws (for $m \geq 1$) defined over analyzers $\mathcal{A}$ by a cohomology theory $(H^n(\mathcal{A}), \delta^n)_{n\geq 1}$.

It is also possible to define a generalized Lazard-Lie theory for complete Hopf algebras instead of complete cogroups, see [Hol01].

We are going to sketch the Lazard-Lie theory for $\mathcal{P}$-Hopf algebras. A complex and cohomology groups are defined using the coalgebra structure of co-addition, in a way which is dual to the Hochschild b'-complex for (not necessarily unitary) $\mathcal{A}s$-algebras (see [Lod]).

The elements of the cohomology group H^1, called pseudo-linear elements by Lazard, are m-tuples of primitive elements in the free $\mathcal{P}$-algebra on m variables.

Definition 4.9.1. Let $X = \{x_1, x_2, \ldots, x_m\}$, and let $V = V_X$. Let $\mathcal{P}$ be an operad equipped with a coherent unit action. Let $n \geq 1$.

We denote by $\widehat{F}_{\mathcal{P}}^{\otimes n}(x_1, \dots, x_m)$ the complete $\mathcal{P}$-algebra generated by the $m \cdot n$ elements

$$x_i^{\otimes n,p} \quad (\text{for } 1 \leq p \leq n,\ 1 \leq i \leq m)$$

given in Remark (3.2.7).

We consider $\widehat{F}_{\mathcal{P}}^{\otimes n}(x_1, \dots, x_m)$ to be multigraded by $(\alpha_1, \alpha_2, \dots, \alpha_n)$ and denote the space of homogeneous elements of total degree $r = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ by $F_{\mathcal{P}}^{\otimes n}(x_1, \dots, x_m)^{(r)}$, i.e.

$$\widehat{F}_{\mathcal{P}}^{\otimes n}(x_1, \dots, x_m) = \prod_{r=1}^{\infty} F_{\mathcal{P}}^{\otimes n}(x_1, \dots, x_m)^{(r)}.$$

On homogeneous elements, α_p is the degree given by

$$\alpha_p(x_i^{\otimes n,q}) = \begin{cases} 1 & : p = q \\ 0 & : \text{else.} \end{cases}$$

The elements of $\widehat{F}_{\mathcal{P}}^{\otimes n}(x_1, \dots, x_m)$ are viewed as (generalized) power series over the "variables" $x_i^{\otimes n,p}$. (For $n = 1$, elements of $\widehat{F}_{\mathcal{P}}^{\otimes 1}(x_1, \dots, x_m)$ are just $\mathcal{P}$-power series in variables $x_1, \dots, x_m$.)

Example 4.9.2. Let $A = K1 \oplus F_{\mathcal{P}}(x_1, \dots, x_m)$ or $A = K1 \oplus \widehat{F}_{\mathcal{P}}(x_1, \dots, x_m)$ be a free $\mathcal{P}$-algebra or a free complete $\mathcal{P}$-algebra.

Then any morphism Δ from A to $A \otimes A$ or $A \hat{\otimes} A$ is given by an m-tuple of elements $\Delta(x_i)$, $i = 1, \dots, m$, of $\widehat{F}_{\mathcal{P}}^{\otimes 2}(x_1, \dots, x_m)$.

If, for all i, $\Delta(x_i) = x_i^{\otimes 2,1} + x_i^{\otimes 2,2} +$ (terms of higher order), then A together with Δ is an augmented $\mathcal{P}$-bialgebra (or a complete $\mathcal{P}$-Hopf algebra in the complete case) if and only if the m elements of $\widehat{F}_{\mathcal{P}}^{\otimes 3}(x_1, \dots, x_m)$ given by

$$\gamma_i := (\Delta \hat{\otimes} \mathrm{id}) \circ \Delta(x_i) - (\Delta \hat{\otimes} \mathrm{id}) \circ \Delta(x_i)$$

(for $i = 1, \dots, m$) are zero.

Remark 4.9.3. The tensor product $\otimes$ can be replaced by the categorical coproduct $\sqcup_{\mathcal{P}}$ of $\mathcal{P}$-algebras in the definition above: One works with (tuples of) $\mathcal{P}$-power series in $m \cdot n$ free variables $x_i^{\sqcup n,p}$. This is the setting of Lazard's work (c.f. [Laz55]) on $\mathcal{P}$-formal group laws (as they are called in [Fre98a], [GK94]).

In the special case where $\mathcal{P} = \mathcal{C}om$, the categorical coproduct $\sqcup_{\mathcal{P}}$ is $\otimes$, and both concepts coincide.

Definition 4.9.4. We define $\widehat{\mathcal{A}}_{\mathcal{P}}^{\otimes n}(x_1, \dots, x_m)$ to be the vector space of m-tuples f with entries f_j in $\widehat{F}_{\mathcal{P}}^{\otimes n}(x_1, \dots, x_m)$ that have the following property:

For all $1 \leq j \leq m$, $1 \leq p \leq n$, every multihomogeneous component g of f_j of total degree ≥ 2 fulfills $\alpha_p(g) \geq 1$. (In other words, if all occurrences of $x_1^{\otimes n,p}, \dots, x_m^{\otimes n,p}$ in g are substituted by 0, the result is 0).

The multigrading is also considered for $\widehat{\mathcal{A}}_{\mathcal{P}}^{\otimes n}(x_1, \dots, x_m)$. We denote the homogeneous part of total degree r by $\mathcal{A}_{\mathcal{P}}^{\otimes n}(x_1, \dots, x_m)^{(r)}$.

Example 4.9.5. Let $A = K1 \oplus \widehat{F}_{\mathcal{P}}(x_1, \ldots, x_m)$ be equipped with a morphism $\Delta : A \to A\hat{\otimes}A$, such that $\Delta(x_i) = x_i^{\otimes 2,1} + x_i^{\otimes 2,2} +$ (terms of higher order), and let γ_i (for $i = 1, \ldots, m$) be defined as in Example (4.9.2).

Then it can be checked that $\gamma := (\gamma_1, \ldots, \gamma_r)$ is an element of $\mathcal{A}_{\mathcal{P}}^{\sqcup 3}(x_1, \ldots, x_m)$.

If $\gamma = 0$, the tuple $f = (\Delta(x_1), \ldots, \Delta(x_m)) \in \mathcal{A}_{\mathcal{P}}^{\sqcup 2}(x_1, \ldots, x_m)$ is called a quantum $\mathcal{P}$-formal group law (compare [Hol], [Hol99]).

Morphisms (coordinate transformations) are given by m-tuples of elements from $\widehat{F}_{\mathcal{P}}(x_1, \ldots, x_m)$, i.e. they are given by elements of $\mathcal{A}_{\mathcal{P}}^{\sqcup 1}(x_1, \ldots, x_m)$. If the coordinate transformation is of the form $x_i \mapsto x_i +$ terms of higher order (all i), one speaks of a strict isomorphism.

Definition 4.9.6. Let Δ_a be the co-addition on $K \oplus \widehat{F}_{\mathcal{P}}(x_1, \ldots, x_m)$ given by $x_i \mapsto x_i \otimes 1 + 1 \otimes x_i, i = 1, \ldots, m$.

Let Δ'_a be given by $\Delta'_a(f) = \Delta_a(f) - f \otimes 1 - 1 \otimes f$.

For $n, i \in \mathbb{N}^*$ with $i < n$, we define $\partial_i : \widehat{F}_{\mathcal{P}}^{\otimes n}(x_1, \ldots, x_m) \to \widehat{F}_{\mathcal{P}}^{\otimes (n+1)}(x_1, \ldots, x_m)$ by

$$\partial_i(f) := \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{i-1} \otimes \Delta'_a \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{n-i}.$$

Defined component-wise on m-tuples, one sets

$$\delta = \delta_n : \mathcal{A}_{\mathcal{P}}^{\otimes n} \to \mathcal{A}_{\mathcal{P}}^{\otimes (n+1)},\ \delta_n := \sum_{i=1}^{n} (-1)^i \partial_i.$$

Remark 4.9.7. One checks that δ restricted to $\mathcal{A}_{\mathcal{P}}^{\otimes n}(x_1, \ldots, x_m)^{(r)}$ takes in fact values in $\mathcal{A}_{\mathcal{P}}^{\otimes (n+1)}(x_1, \ldots, x_m)^{(r)}$ and that $\delta_n \circ \delta_{n-1} = 0$ (all n). In fact, the construction is dual to the Hochschild b'-complex for $\mathcal{A}s$-algebras (see [Lod]).

Definition 4.9.8. Let $\delta_{n,r} := \delta_n | \mathcal{A}_{\mathcal{P}}^{\otimes n}(x_1, \ldots, x_m)^{(r)}$. Let

$$H_r^n = \ker(\delta_{n,r}) / \operatorname{im}(\delta_{n-1,r}).$$

Remark 4.9.9. By the definition of $\mathcal{A}_{\mathcal{P}}^{\otimes n}(x_1, \ldots, x_m)$, $H_r^n = 0$ for all $n > r$ always.

Moreover H_r^r, $r \in \mathbb{N}^*$ is in bijection to anti-symmetric elements of

$$\mathcal{A}_{\mathcal{P}}^{\otimes r}(x_1, \ldots, x_m)^{(r)} = \mathcal{A}_{\mathcal{C}om}^{\otimes r}(x_1, \ldots, x_m)^{(r)}$$

provided $\mathrm{char}(K) = 0$ (see [Laz55], p. 356, and [Hol01], Proposition 5.9).

The torsion theorem (10.1') of [Laz55] says in the categorical case (where $\sqcup$ is used) that $H_r^n = 0$ for all $n \neq r$.

We note that $-\delta_1 = \partial_1 = \Delta'_a$, thus H_r^1 is given by tuples of primitive elements (homogeneous of degree r), and is not zero in general, e.g. for $\mathcal{P} = \mathcal{A}s$ or $\mathcal{P} = \mathcal{M}ag$. Thus there is no "quantum" version of the torsion theorem.

For (quantum) group law r-chunks (i.e. truncated versions of group laws, truncated after the homogeneous terms of degree r) there is the natural question of extendabilty to $(r + 1)$-chunks (and finally to group laws).

Obstructions for the existence of an extension are found in the groups H^3_{r+1}: a 3-chunk extension exists if and only if the given 2-chunk defines a Lie bracket.

Obstructions for the uniqueness are elements of H^2_{r+1}. For example, in the case where $\mathcal{P} = \mathcal{A}s$, the isomorphism classes of m-dimensional 3-chunks with trivial Lie bracket are given by all $(f_1, f_2, \ldots, f_m)$ with $f_n =$

$$x_n^{\otimes 2,1} + x_n^{\otimes 2,2} + \sum_{(j,i,h)\in \underline{m}^3 : i>h} \Psi^{(n)}_{jih}\big([x_h^{\otimes 2,1}, x_i^{\otimes 2,1}]x_j^{\otimes 2,2} + x_j^{\otimes 2,1}[x_i^{\otimes 2,2}, x_h^{\otimes 2,2}]\big), \Psi^{(n)}_{jih} \in K.$$

The homogeneous part of degree 3 is in H^2_3. For a general normal form of such 3-chunks, see [Hol], [Hol99].

It would be interesting to compute the corresponding obstructions and normal forms for $\mathcal{P} = \mathcal{M}ag$.

Remark 4.9.10. Instead of $\mathcal{P} = \mathcal{M}ag$ we can also study the operad $\mathcal{C}mg$. Primitive elements of $\mathcal{M}ag$ form the non-associative Hausdorff series $H(x, y)$ given by $\log(\vee^2(\exp(x), \exp(y)))$ of [GH03], see [DG03] for the definition of exp and log. As a motivation to study primitive elements of $\mathcal{C}mg$, we can look at the series $\exp(\log(1 + x) + \log(1 + y)) - 1$ in commuting (but non-associative) variables x, y. It starts with the terms

$$x + y + xy - \frac{(x, x, y)_b - (x, y, y)_b}{3} + \ldots$$

There are projectors from Prim$\mathcal{M}ag$ to both Prim$\mathcal{C}mg$ and Prim$\mathcal{A}s = \mathcal{L}ie$.

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